Generalized theta functions for holomorphic triples

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Introduction

Let X be a smooth projective curve over an algebraically closed field k of arbitrary characteristic. A triple on X is a triple (E_2, E_1, φ) where E_1, E_2 are vector bundles on X and $\varphi : E_2 \longrightarrow E_1$ is a morphism. For triples, we can introduce the notion of (semi)stability, depending on a real parameter α, and then study the moduli space of α-(semi)stable triples. The properties of triples and their moduli spaces have been investigated in [2] by Bradlow and García-Prada, in [3] by the same authors with Gothen, in [17] by Schmitt in characteristic 0 or in [1] by Álvarez-Cónsul in case of arbitrary characteristic, etc. Particularly, in [2], Bradlow and García-Prada showed that there is a one-to-one correspondence between triples $T = (E_2, E_1, \varphi)$ and the extensions $E_T \in \mathrm{Ext}^1(p^*E_2, p^*E_1 \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-2))$ on $X \times \mathbb{P}^1$ defined by T. By showing that T is α-(semi)stable if and only if E_T is $H(\alpha)$-(semi)stable where $H(\alpha)$ is an ample divisor on $X \times \mathbb{P}^1$, Bradlow and García-Prada constructed the moduli space of α-(semi)stable triples as a closed subscheme of the moduli space of semistable vector bundles of fixed type on $X \times \mathbb{P}^1$. Let $C \in |mH(\alpha)|$ be a smooth projective curve on $X \times \mathbb{P}^1$. It is well known that for m large enough, the restriction of E_T to C is still semistable. We then obtain a morphism from moduli space of α-semistable triples to the moduli space of semistable vector bundles on C which allows us to study triples and their moduli space from moduli space of vector bundles on C.

Let $U_C(r, d)$ be the moduli space of semistable vector bundles of rank r and degree d on C. For a given vector bundle F on C such that $\chi(E \otimes F) = 0$ for some $E \in U_C(r, d)$ (hence for all $E \in U_C(r, d)$), we set $\Theta_F = \{E \in U_C(r, d) | H^0(C, E \otimes F) \neq 0\}$. It was proven by Drezet and Narasimhan in [5] that Θ_F is the underlying space of an effective Cartier divisor, called a generalized theta divisor, if it is a proper subset of $U_C(r, d)$. Moreover, for any vector bundle F' such that $[F'] = n[F]$ in the Grothendieck's group of coherent sheaves $K(C)$, we have $\Theta_{F'} \in |n\Theta_F|$. On smooth projective curve C, it was first proven by Faltings in [8] and then by Seshadri in [18] that a vector bundle E is semistable if and only if there exists a vector bundle F such that $E \otimes F$ is cohomologically trivial, i.e. $H^*(X, E \otimes F) = 0$. We say that F is orthogonal to E. It follows that Θ_F is a generalized theta divisor if F is orthogonal to some $E \in U_C(r, d)$. In this case, the line bundle $\theta_F = \mathcal{O}_{U_C(r,d)}(\Theta_F)$ has a section ζ_F, called a generalized theta function, such that its zero divisor is Θ_F. Studying properties of linear system $|\Theta_F|$: base point freeness, separation properties, ampleness, global generatedness, etc, allows us to construct and study the moduli space of vector bundles on curves by a GIT-free way (see [8], [18], [11], [16], [7], [6], etc). As mentioned above, there is a morphism from moduli space of α-semistable triples on X to the moduli space of semistable vector bundles on C. So there should be a natural way to define and study theta line bundles and theta functions on the moduli space of α-semistable triples.

The aims of this thesis are to extend some properties of α-semistable triples on smooth projective curves to arbitrary characteristic and to add the missing pieces of the study of moduli space of triples: generalized theta line bundles and generalized theta functions. Studying properties of these objects also allows us to define the orthogonal triples and furthermore prepare the ingredients for a GIT- free construction of moduli space of triples.

More in details, we firstly recall in **Chapter** 1 the definitions, basic properties of slope semistability and global generatedness of vector bundles on curves. In particular, we recall the existence of the Harder-Narasimhan filtration for vector bundles and also a modified Harder-Narasimhan filtration which will be used later.

In **Chapter 2**, we recall the numerical equivalence of divisors on ruled surfaces, typically on $X \times \mathbb{P}^1$. We then prove the following Bogomolov's inequality for vector bundles on $X \times \mathbb{P}^1$ in case of arbitrary characteristic,

Theorem 1 (Thm. 2.2.4). *Let E be a vector bundle on $X \times \mathbb{P}^1$ and H be an ample divisor. If E is H-semistable then*

$$\Delta(E) = (\operatorname{rank}(E) - 1)c_1^2(E) - 2\operatorname{rank}(E)c_2(E) \leq 0.$$

As a consequence of Bogomolov's inequality, we obtain then a restriction theorem for semistable vector bundles on $X \times \mathbb{P}^1$ (Cor. 2.3.4).

In **Chapter** 3 we collect the definitions and basic properties of triples, α-semistability of triples and also the existence of the moduli space of triples. For each triple T, we construct a vector bundle E_T on $X \times \mathbb{P}^1$(see Section 3.3). Let p, q be the projections from $X \times \mathbb{P}^1$ onto X and $\mathbb{P}^1$, respectively. Let $\alpha = \frac{a}{b}$ be a positive rational number. Let us fix $H(\alpha) = aF_p + bF_q$ an ample divisor on $X \times \mathbb{P}^1$ where F_p, F_q are any p-fiber and q-fiber, respectively. We give then a proof of the following result of Bradlow and García-Prada in [2] which works in arbitrary characteristic.

Theorem 2 (Thm. 3.3.4). *The triple $T = (E_2, E_1, \varphi)$ is α-(semi)stable if and only if the extension E_T defined by T is $H(\alpha)$-(semi)stable.*

Chapter 4 is the main part of the thesis where we define and study properties of generalized theta line bundles and generalized theta functions on moduli space of triples. Let $\mathcal{T} = (\mathcal{E}_2 \xrightarrow{\varphi_S} \mathcal{E}_1)$ be a family of α-semistable triples on X parametrized by a scheme S. Using the construction of the determinant line bundle associated to a flat family of coherent sheaves (see Section 4.1.1) and the triple product (see Section 4.2.1), we can define the determinant line bundle for triples (Def 4.2.5). For any triple $F = (F_2 \xrightarrow{\psi} F_1)$ on X, we then obtain a line bundle $\theta_{F_2 \longrightarrow F_1}$ on S. In particular, if $\psi : F_2 \longrightarrow F_1$ is surjective and the Euler characteristic of triple product $\mathcal{T}_s \times F$ is zero for all $s \in S$ then $\theta_{F_2 \twoheadrightarrow F_1}$ has a section $\zeta_{F_2 \twoheadrightarrow F_1}$ (see Section 4.2.4). Let $C \in |mH(\alpha)|$ be a smooth projective curve. As an application of Bogomolov's inequality, $E_T|_C$ is still semistable for any $m \geq m_0 = \lceil \frac{1-r}{r}\Delta(E_T) + 1 \rceil$(Cor. 2.3.4). As $E_T|_C$ is semistable, the existence of orthogonal bundles F_C allows us to construct a α-orthogonal triple $F = (F_2 \xrightarrow{\psi} F_1)$(Def 4.2.2) which characterizes the α-semistability of T:

Theorem 3 (Thm. 4.2.3). *Let $T = (E_2 \xrightarrow{\varphi} E_1)$ be a triple on X and $\alpha = \frac{a}{b} \in \mathbb{Q}$. Then T is α-semistable if and only if it has a α-orthogonal triple.*

Let us fix $C \in |m_0 H(\alpha)|$. Let $\mathcal{M}$ be the moduli space of α-semistable triples of type (r_1, r_2, d_1, d_2) on X and $U_C(r, d)$ be the moduli space of semistable vector bundles on C of rank $r = r_1 + r_2$ and degree $d = m_0 \deg_{H(\alpha)}(E_T)$. We have a morphism

$$\mathfrak{t} : \mathcal{M} \longrightarrow U_C(r, d)$$

sending T to $E_T|_C$(Lemma 4.3.1). We define the generalized theta line bundle and generalized theta functions to be the pullback by $\mathfrak{t}$ of the corresponding ones on $U_C(r, d)$(Def. 4.3.2). Let θ_{F_0} be a basic theta divisor on $U_C(r, d)$ and $|p\Theta_{F_0}|$ be a pluritheta linear series. Let $\mathcal{L} = \mathfrak{t}^*\theta_{F_0}$ be a basic theta line bundle on $\mathcal{M}$. Then for each p, we have a generalized theta line bundle $\mathcal{L}^p$ and a linear system $V(\mathcal{L}^p) \subset H^0(\mathcal{M}, \mathcal{L}^p)$ spanned by generalized theta functions. The first and most important applications which we have from the existence of α-orthogonal triples are the base point freeness, ampleness and the universal property of the generalized theta line bundle for triples. Let F_C be a vector bundle on C such that $[F_C] = p[F_0] \in K(C)$. Using F_C we construct a triple $F = (F_2 \overset{\psi}{\twoheadrightarrow} F_1)$(see proof of Lemma 4.2.4 or (4.27)), such that the line bundle $\theta_{F_2 \twoheadrightarrow F_1}$ has a geometric section $\zeta_{F_2 \twoheadrightarrow F_1}$ whose zero divisor is defined by

$$\Theta_{F_2 \twoheadrightarrow F_1} = \{s \in S | F \text{ is not a } \alpha\text{-orthogonal triple of } \mathcal{T}_s\}$$

(see (4.21) and (4.25)). Let $\mathrm{Comp}(p, F)$ be the set of all triples $F' = (F'_2 \overset{\psi'}{\twoheadrightarrow} F'_1)$ such that $[F'_i] = [F_i] \in K(X)$. Then all the line bundles $\theta_{F'_2 \twoheadrightarrow F'_1}$ are isomorphic to $\theta_{F_2 \twoheadrightarrow F_1}$(see Lemma 4.2.6). Therefore we obtain on S a linear system $V(p, F) \subset H^0(S, \theta_{F_2 \twoheadrightarrow F_1})$ spanned by sections $\zeta_{F'_2 \twoheadrightarrow F'_1}$. By the result of Esteves and Popa on the base point freeness of pluritheta linear series in [7], we obtain the following base point freeness of linear systems $V(\mathcal{L}^p)$ and $V(p, F)$.

Corollary 4 (Cor. 4.3.3). *For any $p \geq r^2 + r$, then*

(i) The linear systems $V(\mathcal{L}^p)$ and $V(p, F)$ are base point free.

(ii) There exists a triple $F' = (F_2' \overset{\psi'}{\twoheadrightarrow} F_1') \in \mathrm{Comp}(p, F)$ *such that*

$$\overline{\Theta}_{F_2' \twoheadrightarrow F_1'} = \{T \in \mathcal{M} | F' \textit{ is not a } \alpha\textit{-orthogonal triple of } T\}$$

is a Cartier divisor on $\mathcal{M}$ *and*

$$\mathcal{L}^p \cong \mathcal{O}_{\mathcal{M}}(\overline{\Theta}_{F_2' \twoheadrightarrow F_1'}).$$

Similar to the construction of extensions on $X \times \mathbb{P}^1$ defined by triples on X, for each family $\mathcal{T}$ of α- semistable triples on X parametrized by S, we obtain an induced family $E_{\mathcal{T}}$ of $H(\alpha)$-semistable vector bundles on $X \times \mathbb{P}^1$ and then a family $\mathcal{E} = E_{\mathcal{T}}|_C$ of semistable vector bundles on C parametrized by S. We then have a commutative diagram of induced morphisms

$$\begin{tikzcd} S \arrow[rr, "f_{\mathcal{E}}"] \arrow[dr, "f_{\mathcal{T}}"'] & & U_C(r,d). \\ & \mathcal{M} \arrow[ur, "\mathfrak{t}"'] & \end{tikzcd}$$

By the universal property and the ampleness of the generalized theta line bundle θ_{F_C} on $U_C(r, d)$(Thm. 4.1.2), we obtain the similar properties for $\mathcal{L}^p$.

Corollary 5 (Cor. 4.3.4). *Let* $p \geq 1$ *and* $F'' = (F_2'' \overset{\psi''}{\twoheadrightarrow} F_1'')$ *be any triple in* $\mathrm{Comp}(p, F)$. *Then for any family* $\mathcal{T} = (\mathcal{E}_2 \xrightarrow{\varphi_S} \mathcal{E}_1)$ *of* α*-semistable triples on* X *parametrized by* S, *the line bundle* $\theta_{F_2'' \twoheadrightarrow F_1''}$ *descends to* $\mathcal{L}^p$ *by the induced morphism* $f_{\mathcal{T}} : S \longrightarrow \mathcal{M}$.

Corollary 6 (Cor. 4.3.5). *The line bundle* $\mathcal{L}$ *on* $\mathcal{M}$ *is ample.*

Another important consequence of Theorem 3 is Langton's valuative criterion for triples. Let R be a discrete valuation ring and $X_\eta = \eta \times X$ be the generic fiber of the first projection

$$\mathrm{Spec}(R) \times X \longrightarrow \mathrm{Spec}(R) = \{0, \eta\}.$$

Theorem 7 (Thm. 4.4.4). *Let* T_η *be a* α*-semistable triple of type* (r_1, r_2, d_1, d_2) *on the generic fiber* X_η. *Then there exists a triple* $\mathcal{T}$ *on* X_R *such that* $\mathcal{T}|_{X_\eta} \cong T_\eta$ *and* $T_0 := \mathcal{T}|_{X_0}$ *is* α*-semistable.*

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Chapter 1

Vector bundles on curves

1.1 Semistability of vector bundles

Let X be a smooth projective curve over an algebraically closed field k of arbitrary characteristic. By a vector bundle on X, we mean a locally free coherent sheaf on X. Let E be a nonzero vector bundle on X, $d = \deg(E)$ and $r = \operatorname{rank}(E)$. Then we define the slope of E to be $\mu(E) = \frac{d}{r}$.

Definition 1.1.1. *Let E be a nonzero vector bundle on X. Then E is called (semi)stable if for any nonzero subsheaf $E' \subsetneq E$ we have $\mu(E')(\leq)\mu(E)$.*

Here we use the notation $(\leq)$ introduced by Huybrechts and Lehn in [12, p.11]: If the word "(semi)stable" and the relation sign "$(\leq)$" appear together in a statement, then this statement stands for two assertions, one with "stable" and strict inequality "$<$" and the other with "semistable" and the mild inequality "$\leq$".

Remark 1.1.2. Assume that

$$0 \longrightarrow F \longrightarrow E \longrightarrow G \longrightarrow 0$$

is an exact sequence of vector bundles. Then we have

$$\deg(E) = \deg(F) + \deg(G),$$
$$\operatorname{rank}(E) = \operatorname{rank}(F) + \operatorname{rank}(G).$$

It implies that

$$\mu(F)(\leq)\mu(E) \Longleftrightarrow \mu(E)(\leq)\mu(G).$$

Hence we can also formulate semistability in terms of quotient bundles: E is semistable if and only if for any nonzero quotient G of E, we have $\mu(E) \leq \mu(G)$.

The following properties of semistable vector bundles can be seen easily by the definition.

Lemma 1.1.3. *Let E be a vector bundle on X and L be a line bundle. Then the following holds:*

(i) L is stable.

(ii) E is (semi)stable if and only if $E \otimes L$ is (semi)stable.

(iii) E is (semi)stable if and only if $E^\vee$, its dual bundle, is (semi)stable.

Proposition 1.1.4. *Let E and F be vector bundles on X.*

(i) If E and F are both semistable and $\mu(E) > \mu(F)$ then $\mathrm{Hom}_X(E, F) = 0$.

(ii) If E is stable then it is simple, i.e. $\mathrm{End}(E) \cong k$.

Proof. Let $\phi \in \mathrm{Hom}(E, F)$ be a morphism. If $\phi \neq 0$ then $I = \mathrm{Im}\,\phi \subset F$ is not only a nonzero subsheaf of F but also a quotient of E. Since E and F are semistable, we have

$$\mu(E) \leq \mu(I) \leq \mu(F).$$

This proves (i). For (ii), we show that any endomorphism of E is in form $\lambda\,\mathrm{Id}_E$ for some $\lambda \in k$. Let $\phi \in \mathrm{End}(E)$ be a nontrivial morphism and I be the image of ϕ. As we have shown above, $\mu(E) < \mu(I) < \mu(E)$ if $I \subsetneq E$. This is impossible. Hence ϕ must be surjective and hence an isomorphism. If $\phi \neq \lambda\,\mathrm{Id}_E$ then there exists $x \in X(k)$ such that $\phi_x \in \mathrm{End}_k(E \otimes k(x))$ is not in form $\lambda\,\mathrm{Id}$. Let λ be a eigentvalue of ϕ_x. Then $\phi - \lambda\,\mathrm{Id}_E \in \mathrm{End}(E)$ is nontrivial but not surjective. Hence it is not an isomorphism. This gives us a contradiction. □

For any coherent sheaf E on X, $\chi(E)$ is the Euler characteristic of E, defined by

$$\chi(E) = \dim H^0(X, E) - \dim H^1(X, E).$$

Theorem 1.1.5 (Riemann-Roch formula - [9, Thm. 4.1, p.432])**.** *Let E be a coherent sheaf on smooth projective curve X of genus g_X. Then*

$$\chi(E) = \deg(E) + \mathrm{rank}(E)(1 - g_X).$$

It follows from Riemann-Roch formula that

$$\mu(E) = \frac{\chi(E)}{\mathrm{rank}(E)} + g_X - 1$$

for any nontrivial vector bundle E on X.

Proposition 1.1.6. *Let E and F be vector bundles on X. If $F \neq 0$ and $H^*(E \otimes F) = 0$, i.e. $H^0(X,E) = H^1(X,E) = 0$, then E is semistable.*

Proof. Assume that $F \neq 0$ and $H^*(X, E\otimes F) = 0$. Since $H^*(X, E\otimes F) = 0$, $\chi(E\otimes F) = 0$. It implies that $g_X - 1 = \mu(E \otimes F)$. If E is not semistable then there exists a nontrivial subbundle $G \subsetneq E$ such that $\mu(G) > \mu(E)$. It follows that

$$\mu(G \otimes F) > \mu(E \otimes F) = g_X - 1$$

and hence $\chi(G \otimes F) > 0$. But we also have

$$\chi(G \otimes F) = \dim H^0(X, G \otimes F) - \dim H^1(X, G \otimes F) \leq 0$$

since $H^0(X, G\otimes F) \subseteq H^0(X, E \otimes F) = 0$. This is a contradiction. So E is semistable. $\square$

Definition 1.1.7. *A vector bundle E on X is called globally generated if the evaluation map*

$$e : H^0(X,E) \otimes \mathcal{O}_X \longrightarrow E$$

is surjective.

Theorem 1.1.8 ([9, Thm. 5.17, p.121; Thm. 5.2, p.228])**.** *Let E be a vector bundle on X. Then there exists a number $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,*

(i) $E(n)$ is globally generated.

(ii) $H^q(X, E(n)) = 0$ for all $q > 0$.

The following properties of globally generated vector bundles are useful later.

Proposition 1.1.9 ([11, Prop. 2.6])**.** *Let E be a globally generated vector bundle of rank r on X. Then we have the following exact sequences*

$$0 \longrightarrow \mathcal{O}_X^{r-1} \longrightarrow E \longrightarrow \det(E) \longrightarrow 0$$

and

$$0 \longrightarrow \det(E)^{-1} \longrightarrow \mathcal{O}_X^{r+1} \longrightarrow E \longrightarrow 0.$$

1.2 The Harder-Narasimhan filtration for vector bundles

In this section we will explain the existence and some elementary properties of the Harder-Narasimhan filtration for vector bundles on a smooth projective curve. Moreover, we can make a slight modification on the index set of the Harder-Narasimhan filtration so that it can be indexed by all of rational numbers.

Definition 1.2.1. *Let E be a vector bundle over a smooth projective curve X. A Harder-Narasimhan filtration for E is a flag*

$$0 = E_0 \subset E_1 \subset E_2 \subset .... \subset E_k = E$$

of subbundles of E having the following two properties:

(i) E_i/E_{i-1} *is semistable for* $i = 1, .., k$,

(ii) $\mu(E_{i+1}/E_i) < \mu(E_i/E_{i-1})$ *for* $i = 1, ..., k-1$.

By Riemann-Roch formula,

$$\mu(E') \leq g_X - 1 + \dim H^0(X, E)$$

for any nonzero subsheaf E' of E. Hence there exists a subsheaf of E which is maximal among the subsheaves of E of maximal slope, i.e. there exists a subsheaf $F \subseteq E$ such that for all $G \subseteq E$, $\mu(G) \leq \mu(F)$ and in case of equality, $G \subseteq F$. In fact, F is semistable and unique. We call it the maximal destabilizing subsheaf. The existence of the maximal destabilizing subsheaf induces the existence of the Harder-Narasimhan filtration for a vector bundle on X.

Theorem 1.2.2 ([12, Thm. 1.3.4])**.** *Let E be a vector bundle over a smooth projective curve X, then there exists a unique Harder-Narasimhan filtration for E.*

The following properties come directly from the definition of the Hader-Narasimhan filtration.

Proposition 1.2.3. *Let E be a vector bundle with the Harder-Narasimhan filtration*

$$0 = E_0 \subset E_1 \subset E_2 \subset .... \subset E_k = E.$$

Then we have

(i) $\mu(E_j/E_i) > \mu(E_t/E_i)$ *for* $i < j < t$.

(ii) $\mu(E_t/E_i) > \mu(E_t/E_j)$ *for* $i < j < t$.

(iii) $\mu(E_1) > \mu(E_2) > ... > \mu(E_k) = \mu(E)$.

Let us set $\mu_i = \mu(E_i/E_{i-1})$ for $i = 2, ..., k-1$ and $\mu_{\max} = \mu(E_1), \mu_{\min} = \mu(E/E_{k-1})$. By a slight modification, we can index the Harder-Narasimhan filtration by all of rational numbers as follows: For each $\rho \in \mathbb{Q}$, we set $E_{(\rho)} = E_i$ where i is the index such that

$$\mu_i \geq \rho > \mu_{i+1}.$$

It follows that

$$E_{(\rho)} = 0 \Longleftrightarrow \rho > \mu_{\max} \text{ and } E_{(\rho)} = E \Longleftrightarrow \rho \leq \mu_{\min}. \tag{1.1}$$

Proposition 1.2.4. *The following properties are true:*

(i) If $\rho > \tau$ then $E_{(\rho)} \subseteq E_{(\tau)}$. If $E_{(\rho)} \subsetneq E_{(\tau)}$ then $\rho > \tau$.

(ii) $\mu_{\min}(E_{(\tau)}) \geq \tau, \mu_{\max}(E/E_{(\tau)}) < \tau$.

(iii) $(E_{(\rho)})_{(\tau)} = (E_{(\tau)})_{(\rho)} = E_{(\max\{\rho,\tau\})}$.

Proof. Assume that E has the following Harder-Narasimhan filtration

$$0 = E_0 \subset E_1 \subset E_2 \subset .... \subset E_k = E.$$

Assume $\rho > \tau$ and $E_{(\tau)} = E_i$. Then $\mu_i \geq \tau > \mu_{i+1}$. It implies that

$$\rho > \mu_i \text{ or } \mu_i \geq \rho > \mu_{i+1}.$$

By definition, $E_{(\rho)} \subseteq E_i = E_{(\tau)}$. If $E_{(\rho)} \subsetneq E_i$ then $E_{(\rho)} \subseteq E_{i-1}$. Therefore

$$\rho \geq \mu_{i-1} > \tau$$

and we have (i).

As $E_{(\tau)} = E_i$, the Harder-Narasimhan filtration of $E_{(\tau)}$ is induced from the one of E,

$$0 = E_0 \subset E_1 \subset E_2 \subset .... \subset E_i.$$

By definition, we have

$$\mu_{i+1} < \tau \leq \mu_i = \mu(E_i/E_{i-1}) = \mu_{\min}(E_{(\tau)}).$$

Similarly,

$$0 = E_i/E_i \subset E_{i+1}/E_i \subset E_{i+2}/E_i \subset .... \subset E_k/E_i = E/E_i$$

is the Harder-Narasimhan filtration of $E/E_{(\tau)}$. Then

$$\mu_{\max}(E/E_{(\tau)}) = \mu(E_{i+1}/E_i) = \mu_{i+1} < \tau$$

and we get (ii).

If $\rho \leq \tau$ then $(E_{(\tau)})_{(\rho)} = E_{(\tau)}$ since $\rho \leq \mu_{\min}(E_{(\tau)})$ by (1.1) and (ii). If $\rho > \tau$ then

$$E_{(\rho)} = E_{(\max\{\rho,\tau\})} \subseteq E_{(\tau)} \subseteq E$$

by (i). Therefore $E_{(\rho)} = (E_{(\rho)})_{(\rho)} \subseteq (E_{(\tau)})_{(\rho)} \subseteq E_{(\rho)}$. It proves (iii). □

Corollary 1.2.5. *Let $f : E \longrightarrow F$ be a morphism of vector bundles on X. Then for every $\rho \in \mathbb{Q}$, there exists a unique morphism*

$$f_\rho : E_{(\rho)} \longrightarrow F_{(\rho)}$$

making the following diagram commutative

$$\begin{tikzcd} E_{(\rho)} \arrow[r, "f_\rho"] \arrow[d, hook] & F_{(\rho)} \arrow[d, hook] \\ E \arrow[r, "f"] & F. \end{tikzcd}$$

Proof. Assume that E and F have the Harder-Narasimhan filtration

$$0 = E_0 \subset E_1 \subset E_2 \subset .... \subset E_k = E$$

and

$$0 = F_0 \subset F_1 \subset F_2 \subset .... \subset F_{k'} = F,$$

respectively. Let $\rho \in \mathbb{Q}$ and $E_{(\rho)} = E_i \subseteq E$. The claim is trivial if $f(E_{(\rho)}) = 0$. Assume that $f(E_{(\rho)}) \neq 0$. Let j be the minimal index such that $f(E_{(\rho)}) \subseteq F_j$. Let $\tau = \mu(F_j/F_{j-1})$, then $F_{(\tau)} = F_j$ and the composition

$$E_i \xrightarrow{f} F_j \xrightarrow{\pi} F_j/F_{j-1}$$

is not zero. Suppose that $F_{(\tau)} \nsubseteq F_{(\rho)}$, then $\rho > \tau$. By definition, F_j/F_{j-1} is semistable with slope τ and for $1 \leq \nu \leq i$, $E_\nu/E_{\nu-1}$ is semistable with slope $\mu_\nu \geq \rho > \tau$. It is implied by Proposition 1.1.4 that $\mathrm{Hom}(E_\nu/E_{\nu-1}, F_j/F_{j-1}) = 0$ for all $1 \leq \nu \leq i$. It follows that $\mathrm{Hom}(E_i, F_j/F_{j-1}) = 0$. This is a contradiction since $\pi \circ f$ is not zero. $\square$

For each $\tau \in \mathbb{Q}$, we define the quotient

$$\mathrm{gr}_\tau(E) := E_{(\tau)}/(\cup_{\rho>\tau} E_{(\rho)}).$$

It is clear that $\mathrm{gr}_\tau(E) \neq 0$ if and only if $\tau = \mu(E_i/E_{i-1})$ for some $i \neq 0$. In that case,

$$\mathrm{gr}_{\mu(E_i/E_{i-1})}(E) = E_i/E_{i-1}.$$

The following lemma will be used later.

Lemma 1.2.6. *Let E and F be two vector bundles on X with $\mu(E) > \mu(F)$. Then there exists a rational number τ such that*

$$\frac{\mathrm{rank}(E_{(\tau)})}{\mathrm{rank}(E)} > \frac{\mathrm{rank}(F_{(\tau)})}{\mathrm{rank}(F)}.$$

Proof. We set $M = \{\tau \in \mathbb{Q} | \operatorname{rank}(\operatorname{gr}_\tau(E)) + \operatorname{rank}(\operatorname{gr}_\tau(F)) > 0\}$. Then M is a finite set and we could write $M = \{\tau_1, \tau_2, ..., \tau_n\}$ with $\tau_i > \tau_{i+1}$. We define two functions $\operatorname{rank}_E, \operatorname{rank}_F$ from M to $\mathbb{N}$ by setting $\operatorname{rank}_E(\tau_i) = \operatorname{rank}(\operatorname{gr}_{\tau_i}(E))$ and $\operatorname{rank}_F(\tau_i) = \operatorname{rank}(\operatorname{gr}_{\tau_i}(F))$, respectively. Now we have

$$\begin{aligned}\mu(E) &= \frac{\deg(E)}{\operatorname{rank}(E)} = \frac{\sum_{i=1}^n \deg(\operatorname{gr}_{\tau_i}(E))}{\operatorname{rank}(E)} \\ &= \Sigma_{i=1}^n \tau_i \frac{\operatorname{rank}_E(\tau_i)}{\operatorname{rank}(E)} = \Sigma_{i=1}^n (\tau_i - \tau_{i+1}) \Sigma_{j=1}^i \frac{\operatorname{rank}_E(\tau_j)}{\operatorname{rank}(E)},\end{aligned}$$

where we set $\tau_{n+1} := 0$. Therefore we obtain

$$\mu(E) = \Sigma_{i=1}^{n-1} (\tau_i - \tau_{i+1}) \Sigma_{j=1}^i \frac{\operatorname{rank}_E(\tau_j)}{\operatorname{rank}(E)} + \tau_n.$$

The same formula holds for the slope of F. Since $\mu(E) > \mu(F)$,

$$\Sigma_{i=1}^{n-1} (\tau_i - \tau_{i+1}) (\Sigma_{j=1}^i \frac{\operatorname{rank}_E(\tau_j)}{\operatorname{rank}(E)} - \Sigma_{j=1}^i \frac{\operatorname{rank}_F(\tau_j)}{\operatorname{rank}(F)}) > 0.$$

We have $\tau_i > \tau_{i+1}$ for all i. It implies that there exists i such that

$$\Sigma_{j=1}^i \frac{\operatorname{rank}_E(\tau_j)}{\operatorname{rank}(E)} - \Sigma_{j=1}^i \frac{\operatorname{rank}_F(\tau_j)}{\operatorname{rank}(F)} > 0 \Leftrightarrow \frac{\operatorname{rank}(E_{(\tau_i)})}{\operatorname{rank}(E)} > \frac{\operatorname{rank}(F_{(\tau_i)})}{\operatorname{rank}(F)}.$$

$\square$

Chapter 2

Bogomolov's inequality for vector bundles on $X \times \mathbb{P}^1$ in case of arbitrary characteristic

2.1 Numerical equivalence of divisors on ruled surfaces

Definition 2.1.1. *A geometrically ruled surface is a surface Y, together with a surjective morphism $\pi : Y \longrightarrow X$ to a smooth curve X such that the fiber Y_x is isomorphic to $\mathbb{P}^1$ for every point $x \in X$ and π admits a section.*

One of the simplest examples of ruled surfaces which is our main subject is the ruled surface $X \times \mathbb{P}^1$ together with the first projection $p : X \times \mathbb{P}^1 \longrightarrow X$, where X is a smooth projective curve.

Definition 2.1.2. *A divisor D on a surface Y is numerically equivalent to zero, written $D \equiv 0$, if the intersection number $D.E$ is zero for any divisor E. We say that D and E are numerically equivalent, written $D \equiv E$, if $D - E \equiv 0$.*

Let $\operatorname{Num} Y$ be the quotient of $\operatorname{Pic} Y$ by the subgroup of divisor classes numerically equivalent to zero. Then $\operatorname{Num} Y$ is a finitely generated abelian group. The intersection pairing induces a nondegenerate bilinear pairing $\operatorname{Num} Y \times \operatorname{Num} Y \longrightarrow \mathbb{Z}$.

Proposition 2.1.3 ([9, Prop. 2.3, p. 370]). *Let $\pi : Y \longrightarrow X$ be a ruled surface, X_0 be a section and f be any fiber. Then*

$$\operatorname{Pic} Y \cong \mathbb{Z} \oplus \pi^* \operatorname{Pic} X$$

where $\mathbb{Z}$ is generated by X_0. Also

$$\operatorname{Num} Y \cong \mathbb{Z} \oplus \mathbb{Z},$$

generated by X_0, f *and satisfying* $X_0.f = 1, f^2 = 0$.

On our ruled surface $X \times \mathbb{P}^1$, let $p : X \times \mathbb{P}^1 \longrightarrow X$ and $q : X \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ be the projections. We can choose X_0 to be any $q-$fiber F_q and f be any $p-$fiber F_p. It implies that for any divisor class $D \in \mathrm{Num}(X \times \mathbb{P}^1)$, we have $D \equiv aF_p + bF_q$ for some $a, b \in \mathbb{Z}$ where $F_p^2 = F_q^2 = 0, F_p.F_q = 1$. In particular, we have

Proposition 2.1.4 ([9, Prop. 2.20, p. 382])**.** *A divisor* $H \equiv aF_p + bF_q$ *on* $X \times \mathbb{P}^1$ *is ample if and only if* $a, b \in \mathbb{Z}_{>0}$.

Let $H \equiv aF_p + bF_q$ be an ample divisor on $X \times \mathbb{P}^1$, $D \equiv cF_p + dF_q$. Then $H^2 = 2ab > 0$. If $D.H = 0$ then $ad + bc = 0$. It implies that

$$D^2 = 2cd \leq 0. \tag{2.1}$$

We have $D^2H^2 = 4abcd$, $(D.H)^2 = (ad + bc)^2$. Hence

$$(D.H)^2 - D^2H^2 = (ad - bc)^2 \geq 0 \tag{2.2}$$

for any divisor D.

2.2 Bogomolov's inequality

In this section, we will prove Bogomolov's inequality for torsion free coherent sheaves on the ruled surface $X \times \mathbb{P}^1$ in case of arbitrary characteristic. Firstly, we recall the Grothendieck's splitting theorem for vector bundles on $\mathbb{P}^1$.

Theorem 2.2.1 (Grothendieck's spliting theorem - [12, Thm. 1.3.1])**.** *Let* E *be a vector bundle of rank* r *on* $\mathbb{P}^1$. *Then there exists a unique* r*-tuple* $(a_1, a_2, ..., a_r) \in \mathbb{Z}^r$ *with* $a_1 \leq a_2 \leq ... \leq a_r$, *such that*

$$E \cong \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^1}(a_i).$$

Recall that $p : X \times \mathbb{P}^1 \longrightarrow X$ and $q : X \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ are the projections.

Lemma 2.2.2. *Let* E *be a vector bundle of rank* r *on* $X \times \mathbb{P}^1$. *Suppose that for any* p*-fiber* F_p *we have* $E|_{F_p} \cong \mathcal{O}_{\mathbb{P}^1}^{\oplus r}$. *Then there exists a vector bundle* F *on* X *such that* $E \cong p^*F$.

Proof. See [18, Lm. 4.1]. □

Proposition 2.2.3. *Let* E *be a vector bundle on* $X \times \mathbb{P}^1$. *Suppose that*

$$R^1p_*(E \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1)) = 0.$$

Then there exist vector bundles E_0 and E_1 on X such that the following sequence of vector bundles on $X \times \mathbb{P}^1$

$$0 \longrightarrow p^*E_1 \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow p^*E_0 \longrightarrow E \longrightarrow 0 \tag{2.3}$$

is exact.

Proof. Let $x \in X$ be any closed point and E be a vector bundle on $X \times \mathbb{P}^1$ such that $R^1p_*(E \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1)) = 0$. Then E_x, the restriction of E on the fiber of p over x, is a vector bundle on $F_p = x \times \mathbb{P}^1 \cong \mathbb{P}^1$. We have

$$R^1p_*(E \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1)) \otimes k(x) \cong H^1(\mathbb{P}^1, E_x(-1)).$$

So the condition $R^1p_*(E \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1)) = 0$ implies that

$$H^1(\mathbb{P}^1, E_x(-1)) = 0$$

for all $x \in X$. On $\mathbb{P}^1 \times \mathbb{P}^1$ we have the following resolution of diagonal

$$0 \longrightarrow p_1^*\mathcal{O}_{\mathbb{P}^1}(-1) \otimes p_2^*\mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1} \longrightarrow \mathcal{O}_{\Delta_{\mathbb{P}^1}} \longrightarrow 0, \tag{2.4}$$

where p_1, p_2 are the first and second projection from $\mathbb{P}^1 \times \mathbb{P}^1$. Consider the following commutative diagram

$$\begin{tikzcd}
 & X \times \mathbb{P}^1 \arrow[dl, "p"'] \arrow[r, "q"] & \mathbb{P}^1 \\
X & X \times \mathbb{P}^1 \times \mathbb{P}^1 \arrow[u, "p_{13}"] \arrow[d, "p_{12}"] \arrow[r, "p_{23}"'] & \mathbb{P}^1 \times \mathbb{P}^1 \arrow[u, "p_2"'] \arrow[d, "p_1"] \\
 & X \times \mathbb{P}^1 \arrow[ul, "p"] \arrow[r, "q"] & \mathbb{P}^1,
\end{tikzcd}$$

where p_{ij} are the projections. Pulling back the exact sequence (2.4) to $X \times \mathbb{P}^1 \times \mathbb{P}^1$ by p_{23} and using the commutativity of the above diagram, we get a short exact sequence on $X \times \mathbb{P}^1 \times \mathbb{P}^1$,

$$0 \longrightarrow (p_{12})^*q^*\mathcal{O}_{\mathbb{P}^1}(-1) \otimes (p_{13})^*q^*\mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow \mathcal{O}_{X \times \mathbb{P}^1 \times \mathbb{P}^1} \longrightarrow \mathcal{O}_{X \times \Delta_{\mathbb{P}^1}} \longrightarrow 0.$$

Tensoring this exact sequence with $(p_{12})^*E$ and then applying the functor $(p_{13})_*$, we have a long exact sequence

$$\begin{aligned} 0 &\longrightarrow (p_{13})_*[(p_{12})^*(E \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1))] \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow (p_{13})_*(p_{12})^*E \\ &\longrightarrow (p_{13})_*((p_{12})^*E|_{X \times \Delta_{\mathbb{P}^1}}) \longrightarrow R^1(p_{13})_*[(p_{12})^*(E \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1))] \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1). \end{aligned} \tag{2.5}$$

Consider the following commutative diagram,

$$\begin{tikzcd}
X \times \mathbb{P}^1 \times \mathbb{P}^1 \arrow[r, "p_{13}"'] \arrow[d, "p_{12}"'] & X \times \mathbb{P}^1 \arrow[d, "p"] \\
X \times \mathbb{P}^1 \arrow[r, "p"] & X.
\end{tikzcd}$$

Since all the maps in this diagram are flat, we have

$$R^1(p_{13})_*[(p_{12})^*(E \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1))] \cong p^*R^1p_*(E \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1)) = 0.$$

It implies that the sequence

$$0 \longrightarrow (p_{13})_*[(p_{12})^*(E \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1))] \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1)$$

$$\longrightarrow (p_{13})_*(p_{12})^*E \longrightarrow (p_{13})_*((p_{12})^*E|_{X\times\Delta_{\mathbb{P}^1}}) \longrightarrow 0$$

is exact.

We have $(p_{13})_*((p_{12})^*E|_{X\times\Delta_{\mathbb{P}^1}}) \cong E$. To complete the proof, we need to show that

$$(p_{13})_*[(p_{12})^*(E \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1))] \cong p^*E_0$$

and

$$(p_{13})_*(p_{12})^*E \cong p^*E_1$$

for some vector bundles E_0, E_1 on X. By Lemma 2.2.2, it is clear because

$$(p_{13})_*(p_{12}^*)E|_{F_p} \cong \mathcal{O}_{\mathbb{P}^1} \otimes H^0(\mathbb{P}^1, E|_{F_p})$$

and

$$(p_{13})_*[(p_{12})^*(E \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1))]|_{F_p} \cong \mathcal{O}_{\mathbb{P}^1} \otimes H^0(\mathbb{P}^1, E|_{F_p} \otimes \mathcal{O}_{\mathbb{P}^1}(-1))$$

for any F_p be any p-fiber. □

Let H be an ample divisor on $X \times \mathbb{P}^1$ and E be a vector bundle of rank r. We define the H-degree and H-slope of E by setting

$$\deg_H(E) = c_1(E).H \text{ and } \mu_H(E) = \frac{\deg_H(E)}{r}. \tag{2.6}$$

Then E is called H-(semi)stable if for any nonzero subsheaf $F \subseteq E$ we have $\mu_H(F)(\leq)\mu_H(E)$.

Theorem 2.2.4 (Bogomolov's inequality)**.** *Let E be a vector bundle of rank r and H be an ample divisor on $X \times \mathbb{P}^1$. If E is H-semistable then*

$$\Delta(E) = (r-1)c_1^2(E) - 2rc_2(E) \leq 0,$$

where $c_1(E), c_2(E)$ are the first and second Chern classes of E.

Proof. Suppose that E is H-semistable but $\Delta(E) > 0$. Then we will construct a subsheaf $E' \subset E$ which destabilizes E.

I - **Construction of** E'. By Serre's vanishing theorem we can show that for any vector bundle E on $X \times \mathbb{P}^1$, there exists a line bundle L such that $R^1 p_*(E \otimes L) = 0$. Since the semistability of E is unchanged when we tensor E with L and $\Delta(E) = \Delta(E \otimes L)$, by replacing E with $E \otimes L$ if necessary, we can assume that $R^1 p_*(E \otimes q^* \mathcal{O}_{\mathbb{P}^1}(-1)) = 0$. It follows from Proposition 2.2.3 that E has a resolution

$$0 \longrightarrow p^* E_1 \otimes q^* \mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow p^* E_0 \longrightarrow E \longrightarrow 0, \tag{2.7}$$

where E_1, E_0 are vector bundles on X. Assume that E_i has rank r_i and degree d_i for $i = 1, 2$. By computing Chern characters up to numerical equivalence, we have

$$\mathrm{ch}(p^* E_0) \equiv r_0 + d_0 F_p, \mathrm{ch}(p^* E_1) \equiv r_1 + d_1 F_p, \mathrm{ch}(q^* \mathcal{O}_{\mathbb{P}^1}(-1)) \equiv 1 - F_q,$$

$$\mathrm{ch}(p^* E_1 \otimes q^* \mathcal{O}_{\mathbb{P}^1}(-1)) \equiv \mathrm{ch}(p^* E_1)\, \mathrm{ch}(q^* \mathcal{O}_{\mathbb{P}^1}(-1)) \equiv r_1 + d_1 F_p - r_1 F_q - d_1 [pt],$$

$$\mathrm{ch}(E) \equiv \mathrm{ch}(p^* E_0) - \mathrm{ch}(p^* E_1 \otimes q^* \mathcal{O}_{\mathbb{P}^1}(-1)) \equiv r_0 - r_1 + (d_0 - d_1) F_p + r_1 F_q + d_1 [pt].$$

Therefore

$$c_1(E) = \mathrm{ch}_1(E) \equiv (d_0 - d_1) F_p + r_1 F_q,$$

$$c_2(E) = \frac{c_1^2(E) - 2\,\mathrm{ch}_2(E)}{2} \equiv (d_0 r_1 - d_1 r_1 - d_1)[pt].$$

It follows that

$$\Delta(E) = 2(r_0 d_1 - r_1 d_0) = 2 r_0 r_1 (\mu(E_1) - \mu(E_0)).$$

Assume that $\Delta(E) > 0$, then $\mu(E_1) > \mu(E_0)$. Lemma 1.2.6 allows us to choose $\tau \in \mathbb{Q}$ such that

$$\frac{\mathrm{rank}(E_{1(\tau)})}{r_1} > \frac{\mathrm{rank}(E_{0(\tau)})}{r_0}.$$

We have $p^* E_1 \otimes q^* \mathcal{O}_{\mathbb{P}^1}(-1)|_{F_q} \subset p^* E_0|_{F_q}$ for any q-fiber $F_q \cong X$. Then $E_1 \subset E_0$ and hence $E_{1(\tau)} \subset E_{0(\tau)}$ by Corollary 1.2.5. Particularly, $E_{0(\tau)}$ and $E_{1(\tau)}$ are nontrivial. So

$$\frac{\mathrm{rank}(E_{1(\tau)})}{r_1} > \frac{\mathrm{rank}(E_{0(\tau)})}{r_0} \Longleftrightarrow \frac{\mathrm{rank}(E_{1(\tau)})}{\mathrm{rank}(E_{0(\tau)})} > \frac{r_1}{r_0}.$$

We choose such τ that $\frac{\mathrm{rank}(E_{1(\tau)})}{\mathrm{rank}(E_{0(\tau)})}$ becomes maximal. Assume that $\mathrm{rank}(E_{1(\tau)}) = r_{1\tau}$ and $\mathrm{rank}(E_{0(\tau)}) = r_{0\tau}$. If $\mu(E_{1(\tau)}) > \mu(E_{0(\tau)})$, it follows from Lemma 1.2.6 that there exists $\rho \in \mathbb{Q}$ such that

$$\frac{\mathrm{rank}((E_{1(\tau)})_{(\rho)})}{r_{1\tau}} > \frac{\mathrm{rank}((E_{0(\tau)})_{(\rho)})}{r_{0\tau}}$$

where $(E_{1(\tau)})_{(\rho)}$ and $(E_{0(\tau)})_{(\rho)}$ are both nontrivial. It follows that

$$\frac{\mathrm{rank}((E_{1(\tau)})_{(\rho)})}{\mathrm{rank}((E_{0(\tau)})_{(\rho)})} > \frac{r_{1\tau}}{r_{0\tau}} > \frac{r_1}{r_0}.$$

This contradicts the choice of τ. Hence

$$\mu(E_{1(\tau)}) \leq \mu(E_{0(\tau)}). \tag{2.8}$$

Since $E_{1(\tau)}$ is subbundle of $E_{0(\tau)}$, the image of the composition

$$p^*E_{1(\tau)} \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1) \hookrightarrow p^*E_1 \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1) \hookrightarrow p^*E_0$$

is contained in $p^*E_{0(\tau)}$. We obtain the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & p^*E_{1(\tau)} \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1) \arrow[r] \arrow[d] & p^*E_{0(\tau)} \arrow[r] \arrow[d] & E'' \arrow[r] \arrow[d, "\varphi"] & 0\,. \\
0 \arrow[r] & p^*E_1 \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1) \arrow[r] & p^*E_0 \arrow[r] & E \arrow[r] & 0
\end{tikzcd}$$

Let $E' = \mathrm{Im}(\varphi)$ and $K = \mathrm{Ker}(\varphi)$. We have $\mathcal{O}_{\mathbb{P}^1}^{r_{0\tau}} \cong p^*E_{0(\tau)}|_{F_p} \twoheadrightarrow E''|_{F_p}$, equivalently, $E''|_{F_p}$ is globally generated. Since $K \subset p^*(E_1/E_{1(\tau)}) \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1), H^0(\mathbb{P}^1, K|_{F_p}) = 0$. Therefore $K \neq E''$ and then $E' \neq 0$.

II - E' **destabilizes** E**.** It follows from Proposition 2.1.4 that any ample divisor H on $X \times \mathbb{P}^1$ is in form $aF_p + bF_q$ for some $a, b \in \mathbb{Z}_{>0}$. Suppose that

$$c_1(E') \equiv m'F_p + n'F_q, c_1(E) = mF_p + nF_q,$$

then $\mu_H(E') = \frac{c_1(E').H}{r'} = \frac{m'b+n'a}{r'}$ and $\mu_H(E) = \frac{mb+na}{r}$. We have

$$\deg(E'|_{F_p}) = n', \deg(E'|_{F_q}) = m'.$$

If $\mu(E'|_{F_p}) > \mu(E|_{F_p})$ and $\mu(E'|_{F_q}) > \mu(E|_{F_q})$, then $m'/r' > m/r$ and $n'/r' > n/r$. It follows that

$$\mu_H(E') = \frac{an' + bm'}{r'} > \frac{am + bn}{r} = \mu_H(E).$$

Therefore, to show that E' destabilizes E, it is suffice to show that $E'|_{F_p}$ destabilizes $E|_{F_p}$ and $E'|_{F_q}$ destabilizes $E|_{F_q}$ for any F_p and F_q.

+ $\mu(E'|_{F_p}) > \mu(E|_{F_p})$: Let F_p be any p-fiber. Restricting the resolution (2.7) of E to F_p, we have an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}^{r_1}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^1}^{r_0} \longrightarrow E|_{F_p} \longrightarrow 0.$$

It implies that

$$\mathrm{rank}(E|_{F_p}) = r_0 - r_1, \deg(E|_{F_p}) = r_1.$$

Similarly,

$$\mathrm{rank}(E''|_{F_p}) = r_{0\tau} - r_{1\tau}, \deg(E''|_{F_p}) = r_{1\tau}.$$

As we have seen, $\frac{r_{1\tau}}{r_{0\tau}} > \frac{r_1}{r_0}$, so

$$\mu(E''|_{F_p}) = \frac{r_{1\tau}}{r_{0\tau} - r_{1\tau}} > \mu(E|_{F_p}) = \frac{r_1}{r_0 - r_1}.$$

Restricting the following exact sequence

$$0 \longrightarrow K \longrightarrow E'' \longrightarrow E' \longrightarrow 0 \tag{2.9}$$

to F_p, we get an exact sequence

$$0 \longrightarrow K|_{F_p} \longrightarrow E''|_{F_p} \longrightarrow E'|_{F_p} \longrightarrow 0. \tag{2.10}$$

We have $\mu(E''|_{F_p}) = \frac{r_{1\tau}}{r_{0\tau}-r_{1\tau}} > 0$ by the choice of τ. Since

$$K \subset p^*(E_1/E_{1(\tau)}) \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1)$$

then $K|_{F_p} \subset \mathcal{O}_{\mathbb{P}^1}^{r_1 - r_{1\tau}}$. It follows from Theorem 2.2.1 that

$$\mu(K|_{F_p}) \leq -1 < \mu(E''|_{F_p}).$$

From the exactness of (2.10),

$$\mu(K|_{F_p}) < \mu(E''|_{F_p}) \Longrightarrow \mu(E'|_{F_p}) > \mu(E''|_{F_p}) > \mu(E|_{F_p}).$$

+ $\mu(E'|_{F_q}) > \mu(E|_{F_q})$: Let F_q be any q-fiber. Restricting (2.9) to F_q, we have an exact sequence

$$0 \longrightarrow K|_{F_q} \longrightarrow E''|_{F_q} \longrightarrow E'|_{F_q} \longrightarrow 0 \tag{2.11}$$

where $K|_{F_q} \subseteq E_1/E_{1(\tau)}$. By (iv) of Proposition 1.2.4, $\mu_{\max}(E_1/E_{1(\tau)}) < \tau$, hence

$$\mu(K|_{F_q}) \leq \mu_{\max}(E_1/E_{1(\tau)}) < \tau.$$

Consider the following exact sequence

$$0 \longrightarrow E_{1(\tau)} \longrightarrow E_{0(\tau)} \longrightarrow E''|_{F_q} \longrightarrow 0.$$

Since $\mu(E_{0(\tau)}) \geq \mu(E_{1(\tau)})$ (see (2.8)),

$$\mu(E''|_{F_q}) \geq \mu(E_{0(\tau)}) \geq \tau.$$

Hence $\mu(K|_{F_q}) < \mu(E''|_{F_q})$. It follows from the sequence (2.11) that

$$\mu(E'|_{F_q}) > \mu(E''|_{F_q}).$$

Similarly, since $\mu(E_1) > \mu(E_0)$ and the sequence

$$0 \longrightarrow E_1 \longrightarrow E_0 \longrightarrow E|_{F_q} \longrightarrow 0$$

is exact, we have

$$\mu(E_0) > \mu(E|_{F_q}).$$

It is clear by definition that $\mu(E_{0(\tau)}) \geq \mu(E_0)$. So we have

$$\mu(E'|_{F_q}) > \mu(E''|_{F_q}) \geq \mu(E_{0(\tau)}) \geq \mu(E_0) > \mu(E|_{F_q}).$$

□

Remark 2.2.5. Let E be a torsion free coherent sheaf on $X \times \mathbb{P}^1$ and $E^\vee = \mathrm{Hom}_{\mathcal{O}_X}(E, \mathcal{O}_X)$ be the dual sheaf of E. Then $E^{\vee\vee} = (E^\vee)^\vee$ is a vector bundle of the same rank and $T := E^{\vee\vee}/E$ is a sheaf of rank zero. Consider the following exact sequence

$$0 \longrightarrow E \longrightarrow E^{\vee\vee} \longrightarrow T \longrightarrow 0$$

It is easy to see that $\Delta(E) = \Delta(E^{\vee\vee}) - 2\,\mathrm{rank}(E)\,\mathrm{length}(T) \leq \Delta(E^{\vee\vee})$. Moreover, we have known that E is H-semistable if and only if $E^{\vee\vee}$ is H-semistable. Therefore, Bogomolov's inequality remains true for torsion free coherent sheaves.

2.3 Restriction theorems

Using Bogomolov's inequality, we can consider the semistability of vector bundles on $X \times \mathbb{P}^1$ restricted to a smooth curve.

Lemma 2.3.1 (cf. [13, Thm. 5.1]). *Let $0 = E_0 \subset E_1 \subset ... \subset E_m = E$ be the Harder-Narasimhan filtration of E. Set $F_i = E_i/E_{i-1}, r_i = \mathrm{rank}(F_i), \mu_i = \mu(F_i)$. Then we have*

$$\frac{\Delta(E)}{r} = \sum_{i=1}^{m} \frac{\Delta(F_i)}{r_i} + \frac{1}{r}\sum_{i<j} r_i r_j \left(\frac{c_1(F_i)}{r_i} - \frac{c_1(F_j)}{r_j}\right)^2. \tag{2.12}$$

Proof. We have

$$\begin{aligned}\Delta(E) &= (r-1)c_1^2(E) - 2rc_2(E)\\ &= r(c_1^2(E) - 2c_2(E)) - c_1^2(E)\\ &= 2\,\mathrm{ch}_0(E)\,\mathrm{ch}_2(E) - \mathrm{ch}_1^2(E),\end{aligned}$$

where $\mathrm{ch}_i(E)$ is the i^{th} term in the expression of the Chern character of E:

$$\mathrm{ch}(E) = r + c_1(E) + \frac{(c_1^2(E) - 2c_2(E))}{2}.$$

Since Chern character is additive with respect to an exact sequence, we have

$$\frac{\Delta(E)}{r} = 2\,\mathrm{ch}_2(E) - \frac{c_1^2(E)}{r} = 2\sum_{i=1}^{m}\mathrm{ch}_2(F_i) - \frac{(\sum_{i=1}^m c_1(F_i))^2}{r},$$

$$\sum_{i=1}^{m}\frac{\Delta(F_i)}{r_i}=2\sum_{i=1}^{m}\mathrm{ch}_2(F_i)-\sum_{i=1}^{m}\frac{c_1^2(F_i)}{r_i}.$$

Therefore the quality (2.12) is equivalent to

$$\sum_{i=1}^{m}\frac{c_1^2(F_i)}{r_i}=\frac{1}{r}[(\sum_{i=1}^{m}c_1(F_i))^2+\sum_{i<j}r_ir_j(\frac{c_1(F_i)}{r_i}-\frac{c_1(F_j)}{r_j})^2]. \tag{2.13}$$

But it is clear because

$$\begin{aligned}(\sum_{i=1}^{m}c_1(F_i))^2+\sum_{i<j}r_ir_j(\frac{c_1(F_i)}{r_i}-\frac{c_1(F_j)}{r_j})^2&=\sum_{i=1}^{m}c_1^2(F_i)+\sum_{i<j}(\frac{r_jc_1^2(F_i)}{r_i}+\frac{r_ic_1^2(F_j)}{r_j})\\&=\sum_{j=1}^{m}c_1^2(F_j)+\sum_{j=1}^{m}\frac{(\sum_{i\neq j}r_i)c_1^2(F_j)}{r_j}\\&=r\sum_{j=1}^{m}\frac{c_1^2(F_j)}{r_j}.\end{aligned}$$

□

Proposition 2.3.2 (cf. [13, Thm. 5.1])**.** *Let E be any rank r torsion free sheaf on $X\times\mathbb{P}^1$ and H be an ample divisor. Then we have*

$$H^2\cdot\Delta(E)\leq r^2(\mu_{\max}-\mu)(\mu-\mu_{\min}).$$

Proof. Let $0=E_0\subset E_1\subset\ldots\subset E_m=E$ be the Harder-Narasimhan filtration of E. Set $F_i=E_i/E_{i-1},r_i=\mathrm{rank}(F_i),\mu_i=\mu(F_i)$. We have

$$\frac{\Delta(E)}{r}=\sum\frac{\Delta(F_i)}{r_i}+\frac{1}{r}\sum_{i<j}r_ir_j(\frac{c_1(F_i)}{r_i}-\frac{c_1(F_j)}{r_j})^2\leq\frac{1}{r}\sum_{i<j}r_ir_j(\frac{c_1(F_i)}{r_i}-\frac{c_1(F_j)}{r_j})^2$$

since $\Delta(F_i)\leq 0$ for all i. It follows from (2.2) that

$$H^2(\frac{c_1(F_i)}{r_i}-\frac{c_1(F_j)}{r_j})^2\leq((\frac{c_1(F_i)}{r_i}-\frac{c_1(F_j)}{r_j})H)^2.$$

Since H is ample, $H^2>0$. We have

$$\begin{aligned}\frac{\Delta(E)}{r}&\leq\frac{1}{rH^2}\sum_{i<j}r_ir_j((\frac{c_1(F_i)}{r_i}-\frac{c_1(F_j)}{r_j})H)^2\\&=\frac{1}{rH^2}\sum_{i<j}r_ir_j(\mu_i-\mu_j)^2.\end{aligned}$$

By the definition of the Harder-Narasimhan filtration, we have

$$\mu_{\max}=\mu_1>\mu_2>\ldots>\mu_m=\mu_{\min},$$

$$r = \sum_{i=1}^{m} r_i,$$
$$r\mu(E) = \sum_{i=1}^{m} r_i\mu_i.$$

Moreover

$$\begin{aligned}
\sum_{i<j} r_i r_j(\mu_i - \mu_j)^2 &= r(\sum_{i=1}^{m-1}(\sum_{j\le i} r_j(\mu_j - \mu))(\mu_i - \mu_{i+1})) \\
&\le r(\sum_{i=1}^{m-1}(\sum_{j\le i} r_j(\mu_1 - \mu))(\mu_i - \mu_{i+1})) \\
&= r(\mu_1 - \mu)(\sum_{i=1}^{m-1}(\sum_{j\le i} r_j)(\mu_i - \mu_{i+1})) \\
&= r(\mu_{\max} - \mu)(\sum_{i=1}^{m} r_i\mu_i - r\mu_m) \\
&= r^2(\mu_{\max} - \mu)(\mu - \mu_{\min}).
\end{aligned}$$

So we have
$$\frac{\Delta(E)}{r} \le \frac{1}{rH^2}\sum_{i<j} r_i r_j(\mu_i - \mu_j)^2 \le \frac{r^2(\mu_{\max} - \mu)(\mu - \mu_{\min})}{rH^2}.$$

It implies that
$$H^2\Delta(E) \le r^2(\mu_{\max} - \mu)(\mu - \mu_{\min}).$$

□

Theorem 2.3.3 ([13, Thm. 5.2])**.** *Let E be a vector bundle on $X \times \mathbb{P}^1$ of rank $r \ge 2$. Assume that E is H-stable. Let $C \in |mH|$ be a smooth curve. If $m \ge \frac{1-r}{r}\Delta(E) + 1$ then $E|_C$ is stable.*

Proof. Assume that $E|_C$ is not stable. Then there exists a quotient Q of $E|_C$ which is a vector bundle of rank R and degree d on C such that $\mu(E|_C) \ge \mu(Q)$. Let E' be the kernel of the composition $E \longrightarrow E|_C \longrightarrow Q$ and consider the following exact sequence

$$0 \longrightarrow E' \longrightarrow E \longrightarrow Q \longrightarrow 0,$$

where Q is considered as a sheaf of rank zero on $X \times \mathbb{P}^1$. By computing with Chern character, we have
$$\mathrm{ch}(Q) \equiv RC + (d - \frac{RC^2}{2})[pt].$$

Therefore,

$$\mathrm{ch}(E') = \mathrm{ch}(E) - \mathrm{ch}(Q) \equiv r + (c_1(E) - RmH) + (\frac{\Delta(E) + c_1^2(E)}{2r} - d + \frac{Rm^2H^2}{2})[pt].$$

It follows that

$$\begin{aligned}\Delta(E') &= 2\operatorname{ch}_0(E')\operatorname{ch}_2(E') - \operatorname{ch}_1^2(E') \\ &= \Delta(E) + 2rR(\mu(E|_C) - \mu(Q)) + m^2H^2R(r-R) \\ &\geq \Delta(E) + m^2H^2R(r-R).\end{aligned}$$

If E' is semistable, $\Delta(E') \leq 0$ by Bogomolov's inequality. So we have

$$-\Delta(E) \geq m^2H^2R(r-R) \geq (r-1)m^2H^2$$

since $1 \leq R \leq r-1$. It implies then

$$\frac{1-r}{r}\Delta(E) + 1 \geq \frac{(r-1)^2}{r}m^2H^2 + 1.$$

But it is a contradiction since $r \geq 2, H^2 \geq 1$ and $m \geq \frac{1-r}{r}\Delta(E) + 1$. So E' can not be semistable. In that case, we have

$$\mu_{\max}(E') - \mu(E') = \mu_{\max}(E') - \mu(E) + \frac{R}{r}mH^2 = \frac{R}{r}mH^2 - \frac{dr' - d'r}{rr'},$$

where r', d' are the rank and degree of the maximal destabilizing subbundle of E'. Since E is stable, $dr' - d'r \geq 1$. Moreover $1 \leq r' \leq r-1$, then

$$\mu_{\max}(E') - \mu(E') \leq \frac{R}{r}mH^2 - \frac{1}{r(r-1)}.$$

Consider the following diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & E' \arrow[r] \arrow[d] & S \arrow[d] & \\
0 \arrow[r] & E(-C) \arrow[r] & E \arrow[r] \arrow[d] & E|_C \arrow[r] \arrow[d] & 0 \\
 & & Q \arrow[r, "\mathrm{Id}"] \arrow[d] & Q \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

where S is the kernel of $E|_C \longrightarrow Q$ on C and the rows are exact. By the kernel-cokernel exact sequence,

$$0 \longrightarrow E(-C) \longrightarrow E' \longrightarrow S \longrightarrow 0$$

is an exact sequence where S is a sheaf considered on $X \times \mathbb{P}^1$. In particular $E'^{\vee} \subset E(-C)^{\vee}$ and $E(-C)^{\vee}$ are still stable. Therefore

$$\mu(E') - \mu_{\min}(E') = \mu(E(-C)) + \frac{r-R}{r}mH^2 - \mu_{\min}(E')$$

$$
\begin{aligned}
&= \mu_{\max}(E'^{\vee}) - \mu(E(-C)^{\vee}) + \frac{r-R}{r} mH^2 \\
&\leq \frac{r-R}{r} mH^2 - \frac{1}{r(r-1)}.
\end{aligned}
$$

Now we apply Proposition 2.3.2 for E' and obtain

$$
\begin{aligned}
0 &\geq H^2\Delta(E') - r^2(\mu_{\max}(E') - \mu(E'))(\mu(E') - \mu_{\min}(E')) \\
&\geq H^2\Delta(E) + m^2H^4R(r-R) - r^2(\frac{R}{r}mH^2 - \frac{1}{r(r-1)})(\frac{r-R}{r}mH^2 - \frac{1}{r(r-1)}) \\
&= H^2\Delta(E) + \frac{rH^2}{r-1}m + \frac{1}{(r-1)^2}.
\end{aligned}
$$

Therefore

$$
\frac{rH^2}{1-r} m \geq H^2\Delta(E) + \frac{1}{(r-1)^2} \Longleftrightarrow m \leq \frac{1-r}{r}\Delta(E) - \frac{1}{r(r-1)H^2}.
$$

This contradicts the assumption that $m \geq \frac{1-r}{r}\Delta(E) + 1$. □

Corollary 2.3.4 (cf. [13, Cor. 5.4]). *Let E be a vector bundle of rank $r \geq 2$ on $X \times \mathbb{P}^1$ and $C \in |mH|$ be a general smooth projective curve. Assume that E is H-semistable. If*

$$
m \geq \frac{1-r}{r}\Delta(E) + 1
$$

then $E|_C$ is semistable.

Proof. Assume that we have a short exact sequence of vector bundles on $X \times \mathbb{P}^1$

$$
0 \longrightarrow E_1 \longrightarrow E \longrightarrow E_2 \longrightarrow 0,
$$

where $\mu_H(E_i) = \mu_H(E)$. Let $C \in |mH|$ be a general smooth projective curve. By restricting to C, the sequence

$$
0 \longrightarrow E_1|_C \longrightarrow E|_C \longrightarrow E_2|_C \longrightarrow 0
$$

is still exact and $\mu(E_1|_C) = \mu(E_2|_C) = \mu(E|_C)$. It is easy to see that $E|_C$ is semistable if $E_i|_C$ is semistable for $i = 1, 2$. By induction on the length of a Jordan-Hölder filtration which depends only on E itself, it is enough to assume that E has a Jordan-Hölder filtration of length 2,

$$
0 = E_0 \subset E_1 \subset E.
$$

Set $E_2 = E/E_1$ and consider the exact sequence

$$
0 \longrightarrow E_1|_C \longrightarrow E|_C \longrightarrow E_2|_C \longrightarrow 0
$$

where $\mu(E_1|_C) = \mu(E_2|_C) = \mu(E|_C)$. By Theorem 2.3.3, $E_i|_C$ are semistable if

$$
m \geq \frac{1-r_i}{r_i}\Delta(E_i) + 1.
$$

As in (2.12), we have

$$\frac{\Delta(E)}{r} = \frac{\Delta(E_1)}{r_1} + \frac{\Delta(E_2)}{r_2} + \frac{r_1 r_2}{r}(\frac{c_1(E_1)}{r_1} - \frac{c_1(E_2)}{r_2})^2.$$

It follows that

$$\begin{aligned}\frac{1-r}{r}\Delta(E)+1 &= \frac{1-r_1}{r_1}\Delta(E_1)+1+\frac{1-r_2}{r_2}\Delta(E_2)\\ &\quad + [(1-r)\frac{r_1 r_2}{r}(\frac{c_1(E_1)}{r_1} - \frac{c_1(E_2)}{r_2})^2 - \frac{r_2}{r_1}\Delta(E_1) - \frac{r_1}{r_2}\Delta(E_2)]\end{aligned}$$

We have $\Delta(E_i) \leq 0$ since E_i are semistable. Moreover $\mu(E_2) = \mu(E_1)$. It follows from (2.1) that

$$(\frac{c_1(E_1)}{r_1} - \frac{c_1(E_2)}{r_2})^2 \leq 0.$$

So we have

$$\begin{aligned}\frac{1-r}{r}\Delta(E)+1 &\geq \frac{1-r_1}{r_1}\Delta(E_1)+1+\frac{1-r_2}{r_2}\Delta(E_2)\\ &\geq \max\{\frac{1-r_1}{r_1}\Delta(E_1)+1, \frac{1-r_2}{r_2}\Delta(E_2)+1\}.\end{aligned}$$

Hence for any

$$m \geq \frac{1-r}{r}\Delta(E)+1,$$

$E_i|_C$ is semistable. Therefore, $E|_C$ is semistable.

□

Chapter 3

Triples and their moduli spaces

3.1 Triples and their stability

Let X be a smooth projective curve over an algebraically closed field k. We consider the first notions of triples and their stability.

Definition 3.1.1. *A triple T on X consists of two vector bundles E_1, E_2 on X, together with a morphism $\varphi : E_2 \longrightarrow E_1$ and it is denoted by $T = (E_2 \xrightarrow{\varphi} E_1)$ or simply $T = (E_2, E_1, \varphi)$.*

Let $T = (E_2, E_1, \varphi)$ and $T' = (E_2', E_1', \varphi')$ be triples on X. Then a morphism of triples, $f : T \longrightarrow T'$, is a pair (f_2, f_1) of morphisms such that the following diagram commutes

$$\begin{tikzcd} E_2 \arrow[r, "\varphi"] \arrow[d, "f_2"] & E_1 \arrow[d, "f_1"] \\ E_2' \arrow[r, "\varphi'"] & E_1'. \end{tikzcd} \tag{3.1}$$

In particular, f is an isomorphism if f_1 and f_2 are both isomorphisms.

If $\operatorname{rank}(E_i) = r_i$ and $\deg(E_i) = d_i$ for $i = 1, 2$, then we say that T is of type (r_1, r_2, d_1, d_2).

To introduce the notion of (semi)stability for triples, we first precise the subtriples of a given one and then define the slope which depends on a real parameter.

Definition 3.1.2. *A triple $T' = (E_2', E_1', \varphi)$ is called a subtriple of T if E_i' is a subbundle of E_i for $i = 1, 2$ and the commutative diagram (3.1) holds for $f = (j_2, j_1)$ where $j_i : E_i' \hookrightarrow E_i$ are the inclusions. If $E_1' = E_2' = 0$ then T' is called trivial subtriple. T' is called proper subtriple if it is nontrivial and $T' \neq T$.*

Example 3.1.3. Let $T = (L, E, \varphi)$ be a triple where L is a line bundle on X. Then any subtriple T' of T is either $T' = (0, E', 0)$ where $E' \subset E$ is any subbundle or $T' = (L, E', \varphi)$ where E' is a subbundle of E such that $\varphi(L) \subset E'$.

Definition 3.1.4. *Let $T=(E_2,E_1,\varphi)$ be a triple of type (r_1,r_2,d_1,d_2). For any real number $\alpha\in\mathbb{R}$, we define*

$$\deg_\alpha(T)=d_1+d_2-2\alpha r_1$$

and

$$\mu_\alpha(T)=\frac{\deg_\alpha(T)}{r_1+r_2}=\frac{d_1+d_2-2\alpha r_1}{r_1+r_2}. \tag{3.2}$$

The triple T is called α-(semi)stable if for all nontrivial subtriples T' of T, $\mu_\alpha(T')(\leq)\mu_\alpha(T)$.

Example 3.1.5. We consider the simplest triples $T=(0,E,0)$ or $T=(E,0,0)$. Then

$$\mu_\alpha(T')(\leq)\mu_\alpha(T)\Longleftrightarrow\mu(E')(\leq)\mu(E)$$

where $E'\subsetneq E$ and $T'=(0,E',0)$ or $T'=(E',0,0)$. Hence the α-(semi)stability of T is in fact equivalent to the (semi)stability of E for any α. From now on, we just consider triples T with E_1 and E_2 are both nontrivial.

Remark 3.1.6. The above definition of α-degree and α-slope slightly differ from the definitions given in [2],[4], etc, where the authors defined $\deg_\alpha(T)=d_1+d_2+\alpha r_2$. This modification does not change the (semi)stability of triples and is useful when we connect the semistability of extension bundles on $X\times\mathbb{P}^1$ defined by T with the semistability of T itself .

Lemma 3.1.7. *For a short exact sequence of triples on X,*

$$\begin{tikzcd}
0 \arrow[r] & E_2' \arrow[r] \arrow[d, "\varphi'"] & E_2 \arrow[r] \arrow[d, "\varphi"] & E_2'' \arrow[r] \arrow[d, "\varphi''"] & 0 \\
0 \arrow[r] & E_1' \arrow[r] & E_1 \arrow[r] & E_1'' \arrow[r] & 0,
\end{tikzcd} \tag{3.3}$$

we have

$$\mu_\alpha((E_2',E_1',\varphi'))(\leq)\mu_\alpha((E_2,E_1,\varphi))\Longleftrightarrow\mu_\alpha((E_2,E_1,\varphi))(\leq)\mu_\alpha((E_2'',E_1'',\varphi'')).$$

Proof. Since the rows are exact, we have

$$d_i=d_i'+d_i'',r_i=r_i'+r_i''$$

for $i=1,2$. It follows that

$$\mu_\alpha((E_2,E_1,\varphi))=\frac{\deg_\alpha(E_2,E_1,\varphi)}{r_1+r_2}=\frac{\deg_\alpha(E_2',E_1',\varphi')+\deg_\alpha(E_2'',E_1'',\varphi')}{(r_1'+r_2')+(r_1''+r_2'')}.$$

Hence

$$\frac{\deg_\alpha(E_2,E_1,\varphi)}{r_1+r_2}(\geq)\frac{\deg_\alpha(E_2',E_1',\varphi')}{r_1'+r_2'}\Longleftrightarrow\frac{\deg_\alpha(E_2,E_1,\varphi)}{r_1+r_2}(\leq)\frac{\deg_\alpha(E_2'',E_1'',\varphi')}{r_1''+r_2''}.$$

□

Let $T = (E_2, E_1, \varphi)$ be a triple and L be a line bundle on X. We define the triple $T \otimes L$ to be the triple $(E_2 \otimes L, E_1 \otimes L, \varphi \otimes \mathrm{Id}_L)$.

Lemma 3.1.8. *Let $T = (E_2, E_1, \varphi)$ be a triple on X and L a line bundle. Then T is α-semistable if and only if $T \otimes L$ is α-semistable.*

Proof. We have

$$\mu_\alpha(T \otimes L) = \mu_\alpha(T) + \deg(L).$$

Moreover, any subtriple of $T \otimes L = (E_2 \otimes L, E_1 \otimes L, \varphi \otimes Id_L)$ is induced from a subtriple of T by tensoring with L. Hence T is α-semistable if and only if $T \otimes L$ is. □

It is well known that any stable vector bundle E is simple, i.e. $\operatorname{End} E \cong k$. For stable triples, an analogous result holds true.

Definition 3.1.9. *Let $T = (E_2, E_1, \varphi)$ be a triple. Let us set*

$$\operatorname{End}(T) = \{(u, v) \in \operatorname{End}(E_2) \oplus \operatorname{End}(E_1) | \varphi u = v\varphi\}.$$

We say that T is simple if $\operatorname{End}(T) \cong k$, i.e. the only elements in $\operatorname{End}(T)$ are of the form $(\lambda \,\mathrm{Id}_{E_2}, \lambda \,\mathrm{Id}_{E_1})$ for some $\lambda \in k$.

Proposition 3.1.10 ([2, Prop. 3.10])**.** *Let $T = (E_2, E_1, \varphi)$ be a α-stable triple and $(u, v) \in \operatorname{End}(T)$. Then either (u, v) is trivial or both u, v are isomorphisms.*

Proof. Suppose that u and v are both neither nontrivial nor isomorphism. Then

$$T' = (\operatorname{Ker} u, \operatorname{Ker} v, \varphi) \text{ and } T'' = (\operatorname{Im} u, \operatorname{Im} v, \varphi)$$

are nontrivial subtriples of T. T is α-stable, then

$$\mu_\alpha(T') < \mu_\alpha(T) \text{ and } \mu_\alpha(T'') < \mu_\alpha(T). \tag{3.4}$$

We have an exact sequence of triples

$$0 \longrightarrow T' \longrightarrow T \longrightarrow T'' \longrightarrow 0.$$

It follows from lemma 3.1.7 that $\mu_\alpha(T'') > \mu_\alpha(T)$ if $\mu_\alpha(T) < \mu_\alpha(T')$. This contradicts (3.4). □

Corollary 3.1.11 ([2, Cor. 3.12])**.** *If T is α-stable, then it is simple.*

For any triple $T = (E_2, E_1, \varphi)$ there is always a dual triple $T^\vee = (E_1^\vee, E_2^\vee, \varphi^\vee)$, where $\varphi^\vee$ is the transpose of φ, i.e. the image of φ via the canonical isomorphism

$$\operatorname{Hom}(E_2, E_1) \cong \operatorname{Hom}(E_1^\vee, E_2^\vee).$$

From the one-to-one correspondence between subbundles of a given bundle E and subbundles of $E^\vee$, it follows that there is also a one-to-one correspondence between subtriples of T and subtriples of $T^\vee$. Therefore the α-(semi)stability of T is related to that of $T^\vee$.

Proposition 3.1.12 ([2, Prop. 3.16]). *The triple T is α-(semi)stable if and only if its dual triple $T^\vee$ is α-(semi)stable*

Proof. By symmetric property, it is enough to show that T is α-(semi)stable if $T^\vee$ is. Let $T' = (E_2', E_1', \varphi')$ be nontrivial subtriple of T and T'' be the quotient of T by T'. Then we have a short exact sequence of triples,

$$0 \longrightarrow (T'')^\vee \longrightarrow T^\vee \longrightarrow (T')^\vee \longrightarrow 0.$$

As $T^\vee$ is α-(semi)stable, we have $\mu_\alpha(T^\vee)(\geq)\mu_\alpha((T'')^\vee)$. It follows from Lemma 3.1.7 that $\mu_\alpha(T^\vee)(\leq)\mu_\alpha((T')^\vee)$. But we have

$$\begin{aligned}
\mu_\alpha(T^\vee)(\leq)\mu_\alpha((T')^\vee) &\iff \frac{d_1+d_2+2\alpha r_2}{r_1+r_2}(\geq)\frac{d_1'+d_2'+2\alpha r_2'}{r_1'+r_2'} \\
&\iff \frac{d_1+d_2-2\alpha r_1+2\alpha(r_1+r_2)}{r_1+r_2}(\geq)\frac{d_1'+d_2'-2\alpha r_1'+2\alpha(r_1'+r_2')}{r_1'+r_2'} \\
&\iff \mu_\alpha(T)(\geq)\mu_\alpha(T').
\end{aligned}$$

Hence T is α-(semi)stable. □

3.2 Constraints on the parameter α

The constraints on α for α-(semi)stability of triples were first given by Bradlow and García-Prada in [2] where the authors defined the α-degree of $T = (E_2, E_1, \varphi)$ to be $d_1 + d_2 + \alpha r_2$. Using our definition of the α-degree

$$\deg_\alpha(T) := d_1 + d_2 - 2\alpha r_1,$$

we have,

$$\mu_\alpha(T) = \frac{\deg_\alpha(T)}{r_1+r_2} = \frac{d_1+d_2-2\alpha r_1}{r_1+r_2} = \frac{d_1+d_2+2\alpha r_2}{r_1+r_2} - 2\alpha.$$

It follows that T is α-(semi)stable if and only if T is 2α-(semi)stable by definition of Bradlow and García-Prada. Hence the constraints on α with respect to our definition can be induced by [2, Sec 3.3].

Proposition 3.2.1 (cf. [2, Lm. 3.5]). *Let $T = (E_2, E_1, 0)$ be a triple. Then T is α-semistable if and only if $\alpha = \frac{\mu(E_1)-\mu(E_2)}{2}$ and both bundles E_1, E_2 are semistable. In this case T can not be α-stable.*

Proposition 3.2.2 (cf. [2, Prop. 3.17]). *Let $T = (E_2, E_1, \varphi)$ be a α-(semi)stable triple where $\varphi \neq 0$. Then $\alpha(\geq)\frac{\mu(E_1)-\mu(E_2)}{2}$ and $\alpha(\geq)0$.*

Proposition 3.2.3 (cf. [2, Prop. 3.18]). *Let $T=(E_2,E_1,\varphi)$ be a triple with $r_1\neq r_2$ and $\varphi\neq 0$. If T is α-(semi)stable then*

$$\alpha(\leq)(1+\frac{r_1+r_2}{|r_1-r_2|})\frac{\mu(E_1)-\mu(E_2)}{2}. \tag{3.5}$$

Proof. Let $I=\operatorname{Im}\varphi$ and $K=\operatorname{Ker}\varphi$. Assume that $r_1>r_2$ then φ can not be surjective. Then $T''=(E_2,I,\varphi)$ is proper subtriple of T.

If φ is not injective, $T'=(K,0,\varphi)$ is also nontrivial. As T is α-(semi)stable, we have

$$\mu_\alpha(T')(\leq)\mu_\alpha(T)\Longleftrightarrow r_K(d_1+d_2)-d_K(r_1+r_2)(\geq)2\alpha r_1r_K \tag{3.6}$$

and

$$\mu_\alpha(T'')(\leq)\mu_\alpha(T)\Longleftrightarrow \frac{2d_2-d_K-2\alpha(r_2-r_K)}{2r_2-r_K}(\leq)\frac{d_1+d_2-2\alpha r_1}{r_1+r_2}$$
$$\Longleftrightarrow 2(d_1r_2-d_2r_1)+d_K(r_1+r_2)-r_K(r_1+r_2)(\geq)2\alpha(r_2(r_1-r_2)+r_K(r_1+r_2))-2\alpha_1r_K. \tag{3.7}$$

Let (3.6) add to (3.7), we get

$$d_1r_2-d_2r_1(\geq)\alpha(r_2(r_1-r_2)+r_K(r_1+r_2)).$$

As $\varphi\neq 0$ and T is α-(semi)stable, we have $\alpha(\geq)0$. Therefore

$$d_1r_2-d_2r_1(\geq)\alpha r_2(r_1-r_2) \tag{3.8}$$

since $r_K>0$.

If φ is injective, i.e. $K=0$ then

$$\mu_\alpha(T'')(\leq)\mu_\alpha(T)\Longleftrightarrow d_1r_2-d_2r_1(\geq)\alpha r_2(r_1-r_2).$$

So we always have

$$d_1r_2-d_2r_1(\geq)\alpha(r_2(r_1-r_2)).$$

It implies that

$$\alpha(\leq)(1+\frac{r_1+r_2}{|r_1-r_2|})\frac{(\mu(E_1)-\mu(E_2))}{2}.$$

If $r_1<r_2$ then the dual triple $T^\vee=(E_1^\vee,E_2^\vee,\varphi^\vee)$ is also α -(semi)stable by Propostion 3.1.12. Applying the same computation for $T^\vee$, we get

$$\deg(E_2^\vee)\operatorname{rank}(E_1^\vee)-\deg(E_1^\vee)\operatorname{rank}(E_2^\vee)(\geq)\alpha\operatorname{rank}(E_1^\vee)(\operatorname{rank}(E_2^\vee)-\operatorname{rank}(E_1^\vee))$$
$$\Longleftrightarrow d_1r_2-d_2r_1(\geq)\alpha r_1(r_2-r_1)$$
$$\Longleftrightarrow \alpha(\leq)(1+\frac{r_1+r_2}{|r_1-r_2|})\frac{\mu(E_1)-\mu(E_2)}{2}.$$

□

If $r_1=r_2$ then T could be α-(semi)stable for any $\alpha\geq\frac{\mu(E_1)-\mu(E_2)}{2}$. For example, the triple $T=(\mathcal{O}_X,\mathcal{O}_X,\mathrm{Id}_{\mathcal{O}_X})$, then for any $\alpha(\geq)0$, T is α-(semi)stable.

3.3 The corresponding extensions on $X \times \mathbb{P}^1$ of triples

3.3.1 The construction of extensions

Let $T = (E_2 \xrightarrow{\varphi} E_1)$ be a triple on X and $\alpha = \frac{a}{b} \in \mathbb{Q}$ such that $\gcd(a,b) = 1$. We will construct a vector bundle E_T on $X \times \mathbb{P}^1$ such that its semistability with respect to an ample divisor $H(\alpha)$ is equivalent to the α-semistability of T.

On $\mathbb{P}^1$ we have the Euler sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-2) \longrightarrow H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) \otimes \mathcal{O}_{\mathbb{P}^1}(-1) \xrightarrow{e} \mathcal{O}_{\mathbb{P}^1} \longrightarrow 0. \tag{3.9}$$

Let $p : X \times \mathbb{P}^1 \longrightarrow X$ and $q : X \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ be the projections. By pulling back the exact sequence (3.9) by q and then taking tensor product with p^*E_1, we obtain a short exact sequence

$$0 \longrightarrow p^*E_1 \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-2) \longrightarrow p^*E_1 \otimes V \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1) \xrightarrow{\psi} p^*E_1 \longrightarrow 0 \tag{3.10}$$

where $V := H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$ and $\psi := \mathrm{Id}_{p^*E_1} \otimes q^*e$.

Let us set E_T as the pullback of two morphisms

$$\psi : p^*E_1 \otimes V \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow p^*E_1$$

and

$$p^*\varphi : p^*E_2 \longrightarrow p^*E_1.$$

It is locally given by

$$E_T(U) = \{(\eta, \nu) \in p^*E_1 \otimes V \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1)(U) \oplus p^*E_2(U) | (p^*\varphi)(\nu) = \psi(\eta)\}$$

for any open subset $U \subseteq X \times \mathbb{P}^1$. Since ψ is surjective, we get also a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & p^*E_1 \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-2) \arrow[r] \arrow[d, equal] & E_T \arrow[r] \arrow[d] & p^*E_2 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & p^*E_1 \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-2) \arrow[r] & p^*E_1 \otimes V \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1) \arrow[r] & p^*E_1 \arrow[r] & 0 \end{tikzcd} \tag{3.11}$$

with exact rows. Furthermore, we have an exact sequence

$$0 \longrightarrow E_T \longrightarrow p^*E_1 \otimes V \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1) \oplus p^*E_2 \longrightarrow p^*E_1 \longrightarrow 0. \tag{3.12}$$

Lemma 3.3.1. *Let $T = (E_2 \xrightarrow{\varphi} E_1), T' = (E_2' \xrightarrow{\varphi'} E_1')$ be triples on X. Then any morphism $f : T \longrightarrow T'$ induces a morphism $\bar{f} : E_T \longrightarrow E_{T'}$ of vector bundles on $X \times \mathbb{P}^1$ defined by T and T', respectively. In particular, if f is injective then $\bar{f}$ is.*

***Proof*.** Assume that $f = (f_2, f_1) : T \longrightarrow T'$, i.e. we have a commutative diagram

$$\begin{array}{ccc} E_2 & \xrightarrow{\varphi} & E_1 \\ \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} \\ E_2' & \xrightarrow{\varphi'} & E_1'. \end{array}$$

By the construction of E_T and $E_{T'}$, we have the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E_{T'} & \longrightarrow & p^*E_1' \otimes V \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1) \oplus p^*E_2' & \longrightarrow & p^*E_2' & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & E_T & \longrightarrow & p^*E_1 \otimes V \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1) \oplus p^*E_2 & \longrightarrow & p^*E_2 & \longrightarrow & 0 \end{array}$$

where the square on the right hand side commutes. From the exactness of the rows, there exists a morphism $\bar{f} : E_{T'} \longrightarrow E_T$ making the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E_{T'} & \longrightarrow & p^*E_1' \otimes V \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1) \oplus p^*E_2' & \longrightarrow & p^*E_2' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \bar{f}} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & E_T & \longrightarrow & p^*E_1 \otimes V \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1) \oplus p^*E_2 & \longrightarrow & p^*E_2 & \longrightarrow & 0 \end{array}$$

commutative. It is clear that, $\bar{f}$ is injective if f is injective. □

3.3.2 SL(2)-invariant subbundles

It follows from Lemma 3.3.1 that any subtriple of T defines a subbundle of E_T. But it is obviously not true that every subbundle of E_T is obtained by this way. Hence we would like to look for the subbundles of E_T which are defined by subtriples of T.

The graded ring $S = k[x, y]$ is generated by S_1, the space of homogeneous polynomials of degree 1, as k-algebra. On $\mathbb{P}^1 = \operatorname{Proj} S$, any $s \in S_d$, a homogeneous polynomial of degree d, determines in a natural way a global section of $\mathcal{O}_{\mathbb{P}^1}(d)$. Then x, y determine the standard basic of k-vector space $V = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$. On V, $\mathrm{SL}(2) := \mathrm{SL}_2(V)$ acts naturally by

$$g(Ax + By) = \begin{pmatrix} A \\ B \end{pmatrix}^t g \begin{pmatrix} x \\ y \end{pmatrix}$$

for any $g \in \mathrm{SL}(2)$ and $Ax + By \in V$. We then have a natural action of SL(2) on S_d given by

$$g(s(x, y)) = s(g(x), g(y))$$

for any $s = s(x, y) \in S_d$ and $g \in \mathrm{SL}(2)$. This action is compatible with the action of SL(2) on $\mathbb{P}^1$ given by

$$g \circ P = (du - bv : -cu + av)$$

for any $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2)$ and $P = (u : v) \in \mathbb{P}^1$. Assume that $P = Z(\mathfrak{p}) \in \mathbb{P}^1$, we have

$$\mathcal{O}_{\mathbb{P}^1}(n)_P \cong S(n)_{(\mathfrak{p})}$$

where $S(n)_i = S_{n+i}$ for any $i \geq 0, n \in \mathbb{Z}$ and an isomorphism

$$S(n)_{(\mathfrak{p})} \cong S(n)_{(g(\mathfrak{p}))} \tag{3.13}$$

for any $g \in \mathrm{SL}(2)$. In other words, we get a natural SL(2)-equivariant structure on $\mathcal{O}_{\mathbb{P}^1}(n)$ for any n,

$$\Delta^n : q^*_{\mathrm{SL}(2)}\mathcal{O}_{\mathbb{P}^1}(n) \cong \delta^*\mathcal{O}_{\mathbb{P}^1}(n)$$

locally given by (3.13) where $\delta : \mathrm{SL}(2) \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ is the action and $q_{\mathrm{SL}(2)} : \mathrm{SL}(2) \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ is the second projection.

Let SL(2) act trivially on X, then SL(2) acts on $X \times \mathbb{P}^1$ by

$$g \circ (x, P) = (x, g \circ P)$$

for any $(x, P) \in X \times \mathbb{P}^1$. Under this action, p^*E_2 and p^*E_1 are SL(2)-invariant vector bundles and the morphism

$$p^*\varphi : p^*E_2 \longrightarrow p^*E_1$$

is SL(2)-equivariant.

On $\mathbb{P}^1$ we have an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow V \otimes \mathcal{O}_{\mathbb{P}^1} \xrightarrow{e_0} \mathcal{O}_{\mathbb{P}^1}(1) \longrightarrow 0 \tag{3.14}$$

where e_0 is the evaluation map defined by

$$(e_0)_P(s \otimes \rho) = s_P \cdot \rho.$$

Here s_P is the image of global section $s \in V$ in the stalk $\mathcal{O}_{\mathbb{P}^1}(1)_P, P \in \mathbb{P}^1$ and $\rho \in \mathcal{O}_{\mathbb{P}^1,P}$. Moreover, for any $g \in \mathrm{SL}(2)$, $\Delta^1_{(g,P)}(s_P)$ is the image of the global section $g(s)$ in the stalk $\mathcal{O}_{\mathbb{P}^1}(1)_{g \circ P}$. So we have the following commutative diagram

$$\begin{array}{ccc} V \otimes \mathcal{O}_{\mathbb{P}^1,P} & \longrightarrow & \mathcal{O}_{\mathbb{P}^1}(1)_P \\ \Big\downarrow {\scriptstyle g \otimes \Delta_{(g,P)}} & & \Big\downarrow {\scriptstyle \Delta^1_{(g,P)}} \\ V \otimes \mathcal{O}_{\mathbb{P}^1,g \circ P} & \longrightarrow & \mathcal{O}_{\mathbb{P}^1}(1)_{g \circ P} \end{array}$$

for any $g \in \mathrm{SL}(2)$ and $P \in \mathbb{P}^1$. It implies that e_0 is a SL(2)-equivariant morphism and hence $\mathcal{O}_{\mathbb{P}^1}(-1)$ is SL(2)-invariant. As $e = e_0 \otimes \mathrm{Id}_{\mathcal{O}_{\mathbb{P}^1}(-1)}$, the Euler sequence (3.9) is SL(2)-equivariant. It follows that the morphism

$$\psi = \mathrm{Id}_{p^*E_1} \otimes q^*e : p^*E_1 \otimes V \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow p^*E_1$$

is SL(2)-equivariant. Being the fullback of two SL(2)-equivariant morphisms, $p^*\varphi$ and ψ, E is also SL(2)-invariant.

Proposition 3.3.2. *Let E be a vector bundle on X and $F \subseteq p^*E$ be a subbundle of p^*E on $X \times \mathbb{P}^1$ which is* $\mathrm{SL}(2)$*-invariant. Then $F \cong p^*E'$ for some subbundle $E' \subseteq E$.*

Proof. To prove, we will show that $F|_{F_p} \cong \mathcal{O}_{\mathbb{P}^1}^{\oplus r}$ for any $p-$fiber F_p. Then Lemma 2.2.2 implies that $F \cong p^*E'$ for some $E' \subseteq E$. As a consequence of Proposition 1.1.9, there exists a filtration

$$0 = E_0 \subseteq E_1 \subseteq ... \subseteq E_r = E$$

of subbundles such that E_i/E_{i-1} is a line bundle for each $i = 1, ..., r$. Pulling back this sequence by p gives us a filtration for p^*E,

$$0 = p^*E_0 \subseteq p^*E_1 \subseteq ... \subseteq p^*E_r = p^*E,$$

with the same property. We intersect this sequence with F to get a filtration for F,

$$0 = F_0 \subseteq F_1 \subseteq ... \subseteq F_{r'} = F,$$

where $F_i = p^*E_i \cap F, r' = \mathrm{rank}(F)$ such that F_i/F_{i-1} is a line bundle. For each $i = 1, ..., r'$, we consider the following $\mathrm{SL}(2)$-equivariant diagram

$$\begin{tikzcd}
0 \arrow[r] & F_i \arrow[r] \arrow[d, hook] & F_{i+1} \arrow[r] \arrow[d, hook] & F_{i+1}/F_i \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & p^*E_i \arrow[r] & p^*E_{i+1} \arrow[r] & p^*E_{i+1}/p^*E_i \arrow[r] & 0
\end{tikzcd}$$

Restricting this diagram to $F_p \cong \mathbb{P}^1$, we get

$$\begin{tikzcd}
0 \arrow[r] & F_i|_{F_p} \arrow[r] \arrow[d, hook] & F_{i+1}|_{F_p} \arrow[r] \arrow[d, hook] & \mathcal{O}_{\mathbb{P}^1}(a_i) \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & p^*E_i|_{F_p} \arrow[r] & p^*E_{i+1}|_{F_p} \arrow[r] & \mathcal{O}_{\mathbb{P}^1} \arrow[r] & 0
\end{tikzcd}$$

which is still a $\mathrm{SL}(2)$-equivariant diagram. Consider the $\mathrm{SL}(2)$-equivariant short exact sequences

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(a_i) \longrightarrow \mathcal{O}_{\mathbb{P}^1} \longrightarrow T_i \longrightarrow 0 \tag{3.15}$$

where T_i is torsion sheaf for $i = 1, ..., r'$. Since $\mathcal{O}_{\mathbb{P}^1}$ and $\mathcal{O}_{\mathbb{P}^1}(a_i)$ are $\mathrm{SL}(2)$-invariant, T is also $\mathrm{SL}(2)$-invariant. Moreover, $\mathrm{SL}(2)$ acts transitively on $\mathbb{P}^1$. It follows that $\mathrm{Supp}\, T_i$ is whole $\mathbb{P}^1$ if it is non-empty. This contradicts the fact that T_i is torsion. Hence $T_i = 0$ and $\mathcal{O}_{\mathbb{P}^1}(a_i) \cong \mathcal{O}_{\mathbb{P}^1}$. But $F|_{F_p} \cong \oplus_{i=1}^{r'} \mathcal{O}_{\mathbb{P}^1}(a_i)$, therefore

$$F|_{F_p} \cong \oplus_{i=1}^{r'} \mathcal{O}_{\mathbb{P}^1}.$$

□

We get immediately the following corollary.

Corollary 3.3.3. *Let E_T be the vector bundle on $X \times \mathbb{P}^1$ defined by triple $T = (E_2 \xrightarrow{\varphi} E_1)$ on X and $E' \subset E_T$ be a $\mathrm{SL}(2)$-invariant subbundle. Then E' is defined by a subtriple T' of T*

***Proof*.** Let K and I be the kernel and image of morphism $E' \longrightarrow E_T \longrightarrow p^*E_2$. Then K and I are SL(2)-invariant bundles making the following diagram commutative

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] \arrow[d, hook] & E' \arrow[r] \arrow[d, hook] & I \arrow[r] \arrow[d, hook] & 0 \\ 0 \arrow[r] & p^*E_1 \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-2) \arrow[r] & E_T \arrow[r] & p^*E_2 \arrow[r] & 0. \end{tikzcd} \tag{3.16}$$

By Proposition 3.3.2, $I = p^*E_2'$ and $K = p^*E_1' \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-2)$ for some $E_1' \subset E_1$ and $E_2' \subset E_2$. Taking the pushforward of the diagram (3.16) by p, we get a commutative square

$$\begin{tikzcd} E_2' \arrow[r] \arrow[d, hook] & E_1' \otimes R^1p_*(q^*\mathcal{O}_{\mathbb{P}^1}(-2)) \arrow[d, hook] \\ E_2 \arrow[r] & E_1 \otimes R^1p_*(q^*\mathcal{O}_{\mathbb{P}^1}(-2)). \end{tikzcd}$$

By Serre's duality, we can take two isomorphisms

$$E_1' \otimes R^1p_*(q^*\mathcal{O}_{\mathbb{P}^1}(-2)) \cong E_1',$$

$$E_1 \otimes R^1p_*(q^*\mathcal{O}_{\mathbb{P}^1}(-2)) \cong E_1$$

such that the diagram

$$\begin{tikzcd} E_2' \arrow[r, "\varphi'"] \arrow[d, hook] & E_1' \arrow[d, hook] \\ E_2 \arrow[r, "\varphi"] & E_1 \end{tikzcd}$$

commutes, i.e. $T' = (E_2' \xrightarrow{\varphi'} E_1')$ is a subtriple of T. □

For a given triple $T = (E_2, E_1, \varphi)$ of type (r_1, r_2, d_1, d_2), we set

$$\alpha_m = \frac{\mu(E_1) - \mu(E_2)}{2} \text{ and } \alpha_M = (1 + \frac{r_1 + r_2}{|r_1 - r_2|})\frac{\mu(E_1) - \mu(E_2)}{2} \tag{3.17}$$

if $r_1 \neq r_2$ and $\alpha_M = +\infty$ if $r_1 = r_2$. We have seen from Propositions 3.2.2 and 3.2.3 that T can not be α-semistable unless

$$\alpha_m \leq \alpha \leq \alpha_M \text{ and } \alpha \geq 0.$$

Let $\alpha = \frac{a}{b} \in \mathbb{Q}$ such that $a, b > 0, \gcd(a, b) = 1$ and $\alpha_m \leq \alpha \leq \alpha_M$. Let E_T be the vector bundle on $X \times \mathbb{P}^1$ defined by T as in (3.11) and (3.12). It is obvious that

$$E_T \in \operatorname{Ext}^1(p^* E_2, p^* E_1 \otimes q^* \mathcal{O}_{\mathbb{P}^1}(-2))).$$

On $X \times \mathbb{P}^1$, $H(\alpha) = aF_p + bF_q$ is an ample divisor. We have

$$\deg_{H(\alpha)}(E_T) = b(d_1 + d_2) - 2ar_1$$

and then the $H(\alpha)$-slope of E_T

$$\mu_{H(\alpha)}(E_T) = \frac{b(d_1 + d_2) - 2ar_1}{r_1 + r_2} = b\mu_\alpha(T).$$

Theorem 3.3.4. *The triple $T = (E_2, E_1, \varphi)$ is α-(semi)stable if and only if the extension E_T defined by T is $H(\alpha)$-(semi)stable.*

***Proof*.** Assume that the bundle E_T defined by T is $H(\alpha)$-(semi)stable. Let $T' = (E_2' \xrightarrow{\varphi'} E_1')$ be a nontrivial subtriple of T. It follows from Lemma 3.3.1 that the bundle $E'_{T'}$ defined by T' is a subbundle of E_T. Hence

$$\mu_\alpha(T') = \frac{\mu_{H(\alpha)}(E'_{T'})}{b} (\leq) \frac{\mu_{H(\alpha)}(E_T)}{b} = \mu_\alpha(T),$$

i.e. T is α-(semi)stable.

Assume that T is α-(semi)stable and E_T is not $H(\alpha)$-(semi)stable. Then there exists a maximal destabilizing subsheaf $E' \subseteq E_T$. As E_T is SL(2)-invariant, E' is also SL(2)-invariant because $g^* E' \subseteq E_T$ for every $g \in \mathrm{SL}(2)$ and the action of g does not change the rank E' and the $\deg_{H(\alpha)}(E')$ as SL(2) is connected. By Corollary 3.3.2, $E' = E'_T$ for some subtriple $T' = (E_2' \xrightarrow{\varphi} E_1')$ of T. We have

$$\mu_\alpha(T') = \frac{\mu_{H(\alpha)}(E'_T)}{b} (\geq) \frac{\mu_{H(\alpha)}(E_T)}{b} = \mu_\alpha(T).$$

This contradicts the α-(semi)stability of T. $\square$

3.4 Moduli space of triples

Let S be a scheme of finite type over k and $p : S \times X \longrightarrow S$ be the first projection. A family of triples on X parametrized by S (or a S-family of triples on X) is a triple $\mathcal{T} = (\mathcal{E}_2 \xrightarrow{\varphi_S} \mathcal{E}_1)$ on $S \times X$ where $\mathcal{E}_i$ are families of vector bundles on X parametrized by S. Let $\mathcal{T} = (\mathcal{E}_2 \xrightarrow{\varphi_S} \mathcal{E}_1)$ and $\mathcal{T}' = (\mathcal{E}_2' \xrightarrow{\varphi_S'} \mathcal{E}_1')$ be S-families of triples. We say that they are equivalent if there exists a line bundle L on S and isomorphisms $\psi_i : \mathcal{E}_i \longrightarrow \mathcal{E}_i' \otimes p^* L$ such that

$$\varphi_S = \psi_1^{-1} \circ (\varphi_S' \otimes p^* \operatorname{Id}_L) \circ \psi_2.$$

For a given $\alpha \in \mathbb{Q}_{>0}$ we say that $\mathcal{T} = (\mathcal{E}_2 \xrightarrow{\varphi_S} \mathcal{E}_1)$ is a S-family of α-(semi)stable triples of type $(\underline{r}, \underline{d}) = (r_1, r_2, d_1, d_2)$ if $\mathcal{T}_s = (\mathcal{E}_{2s} \xrightarrow{\varphi_s} \mathcal{E}_{1s})$ is α-(semi)stable and $\mathcal{E}_{is}$ have rank r_i and degree $d_i, i = 1, 2; s \in S$. Let Sch/k and $Sets$ be the category of schemes of finite type over k and the category of sets, respectively. We consider the following contravariant functor:

$$\begin{aligned} \underline{\mathcal{M}}_\alpha^{(s)s}(\underline{r}, \underline{d}) : Sch/k \longrightarrow & Sets \\ S \longmapsto & \{\text{Equivalence classes of } S - \text{families of } \alpha - \text{(semi)stable} \\ & \text{triples of type } (\underline{r}, \underline{d})\}. \end{aligned}$$

There are several approaches to construct the moduli space of α-(semi)stable triples on curves, i.e. the scheme corepresents $\underline{\mathcal{M}}_\alpha^{(s)s}(\underline{r}, \underline{d})$. As we have seen, a triple $T = (E_2 \xrightarrow{\varphi} E_1)$ is α-(semi)stable if and only if the vector bundle E_T on the ruled surface $X \times \mathbb{P}^1$ defined by T is $H(\alpha)$-(semi)stable. It implies that we can embed the functor $\underline{\mathcal{M}}_\alpha^{(s)s}(\underline{r}, \underline{d})$ into the moduli functor of $H(\alpha)$-(semi)stable vector bundles on $X \times \mathbb{P}^1$ which is corepresented by a projective scheme of finite type over k(cf. [13], [18]). By this way, Brallow and García-Prada (cf. [2]) constructed the moduli space of α-stable triples on compact Riemann surfaces.

From another point of view, any triple $T = (E_2 \xrightarrow{\varphi} E_1)$ on X can be seen as a representation of the quiver Q,

$$\bullet \longrightarrow \bullet,$$

consisting of 2 vertices and one arrow, in the category of vector bundles on X. As the semistability of representations is given, we also have the concepts of the Harder-Narasimhan filtration, Jordan-Hölder filtration, the associated graded object for a representation and then the S-equivalence as usual. The moduli space of α-semistable triples is then just a special case of the moduli space of twisted representation of quivers given by A.Schmitt (cf. [17, Thm. 3.7.2]) as the base field k is the field of complex numbers or by Álavrez-Cónsul (cf. [1]) for characteristic p. From there, we obtain the moduli space of α-semistable triples on smooth projective curve X.

Theorem 3.4.1. *There exist a projective scheme $\mathcal{M}_\alpha^{ss}(\underline{r}, \underline{d})$ which corepresents the functor $\underline{\mathcal{M}}_\alpha^{ss}(\underline{r}, \underline{d})$. The closed points of $\mathcal{M}_\alpha^{ss}(\underline{r}, \underline{d})$ are in bijection to the set of S-equivalence classes of α-semistable triples of type (r_1, r_2, d_1, d_2). There is an open subscheme $\mathcal{M}_\alpha^{s}(\underline{r}, \underline{d})$ which corepresents the subfunctor $\underline{\mathcal{M}}_\alpha^{s}(\underline{r}, \underline{d})$ and whose closed points correspond to the isomorphism classes of α-stable triples.*

Proof. See [17, Thm. 3.7.2] or [1, Thm. 8] □

Chapter 4

Generalized theta line bundles

4.1 Theta divisors for vector bundles on curves

4.1.1 Determinant line bundles

Let X be a smooth projective variety of dimension n. We define the Grothendieck's groups $K(X)$ and $K^0(X)$ to be the quotients of free abelian group generated by coherent and locally free sheaves, respectively, by the subgroup generated by all expressions $E-E'-E''$, wherever there is an exact sequence $0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$. Moreover, the tensor product turns $K^0(X)$ into a commutative ring with $1 = [\mathcal{O}_X]$ and gives $K(X)$ a module structure over $K^0(X)$.

Let S be a connected scheme of finite type over k. Then the projection $p : S \times X \longrightarrow S$ is a smooth morphism of relative dimension n. Let $\mathcal{E}$ be a flat family of coherent sheaves on X parametrized by S. We have a homomorphism $p_! : K^0(S \times X) \longrightarrow K^0(S)$, given by

$$p_!([\mathcal{E}]) = \sum_{i=0}^{n} (-1)^i [R^i p_* \mathcal{E}]$$

(cf. [12, Cor. 2.1.11]). Let $q : S \times X \longrightarrow X$ be the second projection. Then we have a well defined morphism

$$\begin{aligned} \lambda_{\mathcal{E}} : K(X) &\longrightarrow \mathrm{Pic}(S) \\ [F] &\mapsto \det(p_!(\mathcal{E} \otimes q^*[F])) \end{aligned} \tag{4.1}$$

which is the composition of morphisms

$$K(X) \xrightarrow{q^*} K^0(S \times X) \xrightarrow{\cdot[\mathcal{E}]} K^0(S \times X) \xrightarrow{p_!} K^0(S) \xrightarrow{\det} \mathrm{Pic}(S).$$

For each class $[F] \in K(X)$, the line bundle $\lambda_{\mathcal{E}}([F])$ is called a determinant line bundle associated to $\mathcal{E}$ at the class $[F]$. The following properties of $\lambda_{\mathcal{E}}$ can be seen easily by using the projection formula for direct images of sheaves and the basic properties of the determinant of locally free sheaves.

Lemma 4.1.1 (cf. [12, Lm. 8.1.2])**.** *Let $\mathcal{E}$ be a S-flat family of coherent sheaves and $[F]$ be any class in $K(X)$.*

(i) (Additive property) If

$$0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}'' \longrightarrow 0$$

is a short exact sequence of S-flat families of coherent sheaves then

$$\lambda_{\mathcal{E}}([F]) = \lambda_{\mathcal{E}'}([F]) \otimes \lambda_{\mathcal{E}''}([F]).$$

Similarly, if we have a short exact sequence of coherent sheaves on X

$$0 \longrightarrow F' \longrightarrow F \longrightarrow F'' \longrightarrow 0,$$

then $\lambda_{\mathcal{E}}([F]) \cong \lambda_{\mathcal{E}}([F']) \otimes \lambda_{\mathcal{E}}([F''])$.

(ii) (Base-change property) If $f : S' \longrightarrow S$ is any morphism, then

$$f^*(\lambda_{\mathcal{E}}([F])) = \lambda_{f_X^*\mathcal{E}}([F])$$

where $f_X := f \times \mathrm{Id}_X : S' \times X \longrightarrow S \times X$.

(iii) (Projection property) If L is a line bundle on S then

$$\lambda_{\mathcal{E}\otimes p^*L}([F]) = \lambda_{\mathcal{E}}([F]) \otimes L^{\chi(\mathcal{E}_s \otimes F)}$$

for any $s \in S$.

4.1.2 Generalized theta functions and generalized theta divisors

Let X be a smooth projective curve and r, d be integers, $r \geq 2$. Let $U_X(r,d)$ be the moduli space of S-equivalence classes of vector bundles on X of rank r and degree d. Then $U_X(r,d)$ is a normal projective variety whose smooth locus is the fine moduli space of stable vector bundles $U_X^s(r,d)$, unless $g = 2, r = 2$ and d is even, when $U_X(r,d)$ is smooth. Drezet and Narasimhan (cf. [5]) even showed that, they are locally factorial schemes. Let F be a nonzero vector bundle on X. We call F a complementary vector bundle of $U_X(r,d)$ if $\chi(E \otimes F) = 0$ for all $E \in U_X(r,d)$. Let $h = \gcd(r,d)$ and $r_0 = \frac{r}{h}, d_0 = \frac{d}{h}$. By Riemann-Roch formula, it is easy to see that F has rank pr_0 and degree $p(r_0(g_X - 1) - d_0)$ for some $p \in \mathbb{Z}_{>0}$. In particular, to a generic vector bundle F_0 of rank r_0 and degree $r_0(g_X-1)-d_0$ we can associate an effective Cartier divisor Θ_{F_0} on $U_X(r,d)$ (cf. [5, Section 0.2]) supported on the set

$$\Theta_{F_0} = \{E \in U_X(r,d) | H^0(E \otimes F) \neq 0\},$$

called a basic theta divisor.

Let S be a connected scheme and $\mathcal{E}$ be a family of vector bundles of rank r and degree d parametrized by S. We will show by the following construction that for each vector bundle F on X such that $\chi(\mathcal{E}_s \otimes F) = 0$ for some $s \in S$ (hence for all $s \in S$), the line bundle $\lambda_{\mathcal{E}}([F])^{-1}$ has a geometric section $\zeta_{\mathcal{E}}(F)$ whose zero divisor is given by

$$\{s \in S | H^0(X, \mathcal{E}_s \otimes F) \neq 0\}. \tag{4.2}$$

Indeed, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$H^0(X, \mathcal{E}_s \otimes \mathcal{O}_X(-n)) = 0$$

for all $s \in S$. Let F be any vector bundle of rank r' on X. Proposition 1.1.9 implies that for n large enough, there exists a short exact sequence of vector bundles on X

$$0 \longrightarrow \det(F)^{-1}(-(1+r')n) \xrightarrow{\varphi_F(n)} \mathcal{O}_X^{r'+1}(-n) \longrightarrow F \longrightarrow 0. \tag{4.3}$$

We consider such large n that it is also bigger than n_0 and denote by $n \gg n_0$. Pulling back this exact sequence by q and then tensoring with $\mathcal{E}$, we get an exact sequence on $S \times X$

$$0 \longrightarrow \mathcal{E} \otimes q^* \det(F)^{-1}(-(1+r')n) \xrightarrow{\mathrm{Id}_{\mathcal{E}} \otimes q^* \varphi_F(n)} \mathcal{E} \otimes q^* \mathcal{O}_X^{r'+1}(-n) \longrightarrow \mathcal{E} \otimes q^* F \longrightarrow 0.$$

Pushing forward to S by p, we have

$$\begin{aligned} 0 \longrightarrow p_*(\mathcal{E} \otimes q^* \det(F)^{-1}(-(r'+1)n)) &\longrightarrow p_*(\mathcal{E} \otimes q^* \mathcal{O}_X^{r'+1}(-n)) \longrightarrow p_*(\mathcal{E} \otimes q^* F) \\ \longrightarrow R^1 p_*(\mathcal{E} \otimes q^* \det(F)^{-1}(-(r'+1)n)) &\xrightarrow{R^1(\varphi_F(n))} R^1 p_*(\mathcal{E} \otimes q^* \mathcal{O}_X^{r'+1}(-n)) \\ &\longrightarrow R^1 p_*(\mathcal{E} \otimes q^* F) \longrightarrow 0 \end{aligned} \tag{4.4}$$

where $R^1(\varphi_F(n)) = R^1 p_*(\mathrm{Id}_{\mathcal{E}} \otimes q^* \varphi_F(n))$. For any $s \in S$, we have

$$p_*(\mathcal{E} \otimes q^* \mathcal{O}_X^{r'+1}(-n)) \otimes k(s) \cong H^0(X, \mathcal{E}_s^{r'+1}(-n)) = 0.$$

Hence the exact sequence (4.4) becomes

$$\begin{aligned} 0 \longrightarrow p_*(\mathcal{E} \otimes q^* F) \longrightarrow R^1 p_*(\mathcal{E} \otimes q^* \det(F)^{-1}(-(r'+1)n)) &\xrightarrow{R^1(\varphi_F(n))} R^1 p_*(\mathcal{E} \otimes q^* \mathcal{O}_X^{r'+1}(-n)) \\ &\longrightarrow R^1 p_*(\mathcal{E} \otimes q^* F) \longrightarrow 0. \end{aligned} \tag{4.5}$$

Then we have

$$\lambda_{\mathcal{E}}([F])^{-1} \cong \det(R^1 p_*(\mathcal{E} \otimes q^* \det(F)^{-1}(-(r'+1)n)))^{-1} \otimes \det(R^1 p_*(\mathcal{E} \otimes q^* \mathcal{O}_X^{r'+1}(-n))). \tag{4.6}$$

Assume that for some $m \gg n_0$, we have another resolution of F of form

$$0 \longrightarrow \det(F)^{-1}(-(1+r')m) \xrightarrow{\varphi_F(m)} \mathcal{O}_X^{r'+1}(-m) \longrightarrow F \longrightarrow 0. \tag{4.7}$$

Then we have the same long exact sequence as (4.5) for m. It implies that

$$\det(R^1p_*(\mathcal{E}\otimes q^*\det(F)^{-1}(-(r'+1)n)))^{-1}\otimes\det(R^1p_*(\mathcal{E}\otimes q^*\mathcal{O}_X^{r'+1}(-n)))$$
$$\cong\det(R^1p_*(\mathcal{E}\otimes q^*\det(F)^{-1}(-(r'+1)m)))^{-1}\otimes\det(R^1p_*(\mathcal{E}\otimes q^*\mathcal{O}_X^{r'+1}(-m))), \tag{4.8}$$

equivalently, $\lambda_{\mathcal{E}}([F])^{-1}$ is independent of the choice of n provided $n\gg n_0$. It is then clear that

$$\lambda_{\mathcal{E}}([F])\cong\lambda_{\mathcal{E}}([F']) \tag{4.9}$$

for any F,F' such that $\mathrm{rank}(F)=\mathrm{rank}(F')$ and $\det(F)=\det(F')$.

Assume that $\chi(\mathcal{E}_s\otimes F)=0$ for some $s\in S$ (hence for all $s\in S$). It follows from Riemann - Roch formula that

$$\mathrm{rank}(p_*(\mathcal{E}\otimes q^*F))=\mathrm{rank}(R^1p_*(\mathcal{E}\otimes q^*F)).$$

From the exactness of (4.5) we have

$$\mathrm{rank}(R^1p_*(\mathcal{E}\otimes q^*\det(F)^{-1}(-(r'+1)n)))=\mathrm{rank}(R^1p_*(\mathcal{E}\otimes q^*\mathcal{O}_X^{r'+1}(-n))).$$

It implies that

$$R^1(\varphi_F(n)):R^1p_*(\mathcal{E}\otimes q^*\det(F)^{-1}(-(r'+1)n))\longrightarrow R^1p_*(\mathcal{E}\otimes q^*\mathcal{O}_X^{r'+1}(-n))$$

is a morphism of vector bundles of the same rank. Therefore the line bundle

$$\lambda_{\mathcal{E}}([F])^{-1}\cong\det(R^1p_*(\mathcal{E}\otimes q^*\det(F)^{-1}(-(r'+1)n)))^{-1}\otimes\det(R^1p_*(\mathcal{E}\otimes q^*\mathcal{O}_X^{r'+1}(-n)))$$

has a geometric section $\zeta_{\mathcal{E}}(F)=\det(R^1(\varphi_F(n)))$. Since $\chi(\mathcal{E}_s\otimes F)=0$ for all $s\in S$, by the construction, we see that the zero divisor of $\zeta_{\mathcal{E}}(F)$ parametrizes all points $s\in S$ such that $H^0(S,\mathcal{E}_s\otimes F)\neq 0$ (cf. [11, Prop. 4.1]).

Assume further that $\mathcal{E}$ is a family of semistable vector bundles of rank r and degree d on X parametrized by S. Then there exists a morphism

$$f_{\mathcal{E}}:S\longrightarrow U_X(r,d), s\mapsto\mathcal{E}_s$$

induced by the family $\mathcal{E}$. Let $[F]\in K(X)$ be any class where F is a complementary vector bundle of $U_X(r,d)$. Then $\chi(\mathcal{E}_s\otimes F)=0$ for all $s\in S$. We obtain on S a line bundle $\lambda_{\mathcal{E}}([F])^{-1}$ and a generalized theta function $\zeta_{\mathcal{E}}(F)$.

Theorem 4.1.2 (cf. [5, Thm. D, part (a)], [15, Cor. 1.6, Cor. 1.7] or [12, Thm. 8.1.5])**.** *Let $[F]$ be a class in $K(X)$ such that F is a complementary vector bundle of $U_X(r,d)$. There exists a unique ample line bundle $\mathbb{L}([F])$ on $U_X(r,d)$ such that for any family $\mathcal{E}$ of semistable vector bundles of rank r and degree d on X parametrized by S, we have*

$$\lambda_{\mathcal{E}}([F])^{-1}\cong f_{\mathcal{E}}^*\mathbb{L}([F]). \tag{4.10}$$

Moreover, the geometric section $\zeta_{\mathcal{E}}(F)$ descends by $f_{\mathcal{E}}$ to a section of $\mathbb{L}([F])$ on $U_X(r,d)$ whose zero divisor is supported on the set

$$\Theta_F = \{E \in U_X(r,d) | H^0(X, E \otimes F) \neq 0\}.$$

We denote this section by ζ_F and call it a generalized theta function. As $\mathbb{L}([F])$ is independent of the choice of representations of class $[F] \in K(X)$, we denote it again by θ_F and call it a generalized theta line bundle. If ζ_F is a nonzero section, i.e. there exists $E \in U_X(r,d)$ such that $H^0(E \otimes F) = 0$, then Θ_F is a divisor, called a generalized theta divisor, and $\theta_F = \mathcal{O}_{U_X(r,d)}(\Theta_F)$. In particular, if we take $F = F_0$ then $\lambda_{\mathcal{E}}([F_0])^{-1}$ descends to basic theta line bundle $\theta_{F_0} = \mathcal{O}_{U_X(r,d)}(\Theta_{F_0})$. Let F_0' be another vector bundle such that $\Theta_{F_0'}$ is a basic theta divisor. We have a morphism of Picard groups $\det^* : \mathrm{Pic}(J^d(X)) \longrightarrow \mathrm{Pic}(U_X(r,d))$ induced from the canonical morphism $\det : U_X(r,d) \longrightarrow J^{(d)}(X)$. As $\mathrm{Pic}^0(X) \subset \mathrm{Pic}(J^d(X))$, we have

$$\theta_{F_0'} \cong \theta_{F_0} \otimes \det^*(\det(F_0') \otimes \det(F_0)^{-1}) \tag{4.11}$$

(cf. [5, Thm. D]).

4.1.3 Pluritheta linear series on $U_X(r,d)$

Proposition 1.1.6 gives us a sufficient condition under which a vector bundle is semistable: E is semistable if there exists a nonzero vector bundle F such that $E \otimes F$ is cohomologically trivial, i.e. $H^*(X, E \otimes F) = 0$. In fact, it was proven by Faltings in [8], Seshadri in [18, Thm. 6.2], etc, that this is also a necessary condition for semistability of vector bundles on curves. In [6, Thm. 2], Esteves even showed that we can find such vector bundle F with fixed determinant. We recall this result for $E \in U_X(r,d)$.

Theorem 4.1.3 ([6, Thm. 2])**.** *Let $E \in U_X(r,d)$ be a semistable vector bundle on X and $\{L_p | p > 0\}$ be a sequence of line bundles such that* $\deg(L_p) = p(r_0(g_X - 1) - d_0)$ *for every integer $p > 0$. Then there is a vector bundle F on X of rank pr_0 such that* $\det(F) \cong L_p$ *and $E \otimes F$ is cohomologically trivial for a certain $p > 0$.*

Let F_0 be a complementary vector bundle of $U_X(r,d)$ such that Θ_{F_0} is a basic theta divisor. For each integer p, let us set $L_p = (\det(F_0))^p$ and consider the complementary vector bundles F of rank pr_0 and determinant L_p. It follows from Theorem 4.1.2 and (4.11) that for any such F, the generalized theta line bundle θ_F is ample and isomorphic to $\theta_{F_0}^p$. In particular, if ζ_F is a nonzero section, then $\Theta_F \in |p\Theta_{F_0}|$. We call $|p\Theta_{F_0}|$ a pluritheta linear series. What we are interested in are the properties of this linear series as base point freeness, separated properties, very ampleness, etc. These properties should be related to a bound on the rank of F.

Theorem 4.1.4 ([7, Thm. B]). *Let Θ_{F_0} be a basic theta divisor on $U_X(r,d)$. For each $p \geq r^2 + r$, the linear series $|p\Theta_{F_0}|$ on $U_X(r,d)$ separates points and is very ample on the smooth locus $U_X^s(r,d)$.*

4.2 Determinant line bundles and invariant sections for triples

4.2.1 Triple product

For any triples $T = (E_2 \xrightarrow{\varphi} E_1)$ and $F = (F_2 \xrightarrow{\psi} F_1)$ on X, we define the triple product of T and F to be the triple

$$T \times F := (E_2 \otimes F_1 \oplus E_1 \otimes F_2 \xrightarrow{(\varphi \otimes \mathrm{Id}_{F_1})\pi_1 - (\mathrm{Id}_{E_1} \otimes \psi)\pi_2} E_1 \otimes F_1) \tag{4.12}$$

where $\pi_1 : E_2 \otimes F_1 \oplus E_1 \otimes F_2 \longrightarrow E_2 \otimes F_1$ and $\pi_2 : E_2 \otimes F_1 \oplus E_1 \otimes F_2 \longrightarrow E_1 \otimes F_2$ are the projections. It is easy to see that the triple product is exact, i.e. if

$$0 \longrightarrow T' \longrightarrow T \longrightarrow T'' \longrightarrow 0 \tag{4.13}$$

is an exact sequence of triples on X then for any triple F, the sequence

$$0 \longrightarrow T' \times F \longrightarrow T \times F \longrightarrow T'' \times F \longrightarrow 0$$

is exact.

Definition 4.2.1. *For a given triple $T = (E_2 \xrightarrow{\varphi} E_1)$ on X, we call $\chi(T)$ the Euler Characteristic of T, defined by*

$$\chi(T) = \chi(E_2) - \chi(E_1),$$

where $\chi(E_i)$ is the Euler Characteristic of $E_i, i = 1, 2$.

For an exact sequence of triples as (4.13), we have

$$\chi(T) = \chi(T') + \chi(T'').$$

4.2.2 α-orthogonal triples

Let $T = (E_2 \xrightarrow{\varphi} E_1)$ and $F = (F_2 \xrightarrow{\psi} F_1)$ be triples on X. Let K be the kernel of

$$\varphi \times \psi := (\varphi \otimes \mathrm{Id}_{F_1})\pi_1 - (\mathrm{Id}_{E_1} \otimes \psi)\pi_2$$

as in (4.12). We then have a short exact sequence

$$0 \longrightarrow K \longrightarrow E_2 \otimes F_1 \oplus E_1 \otimes F_2 \xrightarrow{\varphi \times \psi} E_1 \otimes F_1.$$

Assume that $T = (E_2 \xrightarrow{\varphi} E_1)$ is of type (r_1, r_2, d_1, d_2). As in (3.17), let us set

$$\alpha_m = \frac{\mu(E_1) - \mu(E_2)}{2} \text{ and } \alpha_M = (1 + \frac{r_1 + r_2}{|r_1 - r_2|})\frac{\mu(E_1) - \mu(E_2)}{2}$$

if $r_1 \neq r_1$ and $\alpha_M = +\infty$ if $r_1 = r_1$. From now on, by a rational number $\alpha = \frac{a}{b} \in \mathbb{Q}$, we mean a rational number $\alpha = \frac{a}{b} \in \mathbb{Q}$ such that $a, b > 0, (a, b) = 1$ and $\alpha_m \leq \alpha \leq \alpha_M$.

Definition 4.2.2. *Let $T = (E_2 \xrightarrow{\varphi} E_1), F = (F_2 \xrightarrow{\psi} F_1)$ be triples on X and $\alpha = \frac{a}{b} \in \mathbb{Q}$. Then F is called a α-orthogonal triple of T if it satisfies the following properties:*

(i) ψ *is surjective,*

(ii) $\mathrm{rank}(F_2) = 2\,\mathrm{rank}(F_1)$, $\mu(F_2) = \mu(F_1) - \alpha$,

(iii) $H^*(X, K) = 0$ *where K is the kernel of $\varphi \times \psi$.*

Similar to the semistability of vector bundles, we can characterize the α-semistability of triples by a perpendicularity condition.

Theorem 4.2.3. *Let $T = (E_2 \xrightarrow{\varphi} E_1)$ be a triple of type (r_1, r_2, d_1, d_2) on X and $\alpha = \frac{a}{b} \in \mathbb{Q}$. Then T is α-semistable if and only if it has a α-orthogonal triple.*

***Proof*.** Assume that $F = (F_2 \xrightarrow{\psi} F_1)$ is a α-orthogonal triple of $T = (E_2 \xrightarrow{\varphi} E_1)$. If T is not α-semistable, there exist a subtriple $T' = (E_2' \xrightarrow{\varphi'} E_1')$ which destabilizes T, i.e. $\mu_\alpha(T') > \mu_\alpha(T)$. Consider the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K' \arrow[r] \arrow[d, hook] & E_2' \otimes F_1 \oplus E_1' \otimes F_2 \arrow[r] \arrow[d, hook] & E_1' \otimes F_1 \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & K \arrow[r] & E_2 \otimes F_1 \oplus E_1 \otimes F_2 \arrow[r] & E_1 \otimes F_1 \arrow[r] & 0
\end{tikzcd}$$

where the rows are exact.

Assume that F is of type (R_1, R_2, D_1, D_2). Then

$$R_2 = 2R_1, D_2 = 2D_1 - 2\alpha R_1$$

by condition (ii) in Definition 4.2.2. We have

$$\mathrm{rank}(K) = (r_1 + r_2)R_1 \text{ and } \deg(K) = (r_1 + r_2)D_1 + R_1(d_1 + d_2) - 2\alpha r_1 R_1.$$

It follows that

$$\mu(K) = \mu_\alpha(T) + \mu(F_1). \tag{4.14}$$

Similarly, $\mu(K') = \mu_\alpha(T') + \mu(F_1)$. Since $H^*(X, K) = 0$ then $\chi(K) = 0$. Moreover, $K' \subseteq K$ implies that $\chi(K') \leq 0$. Therefore

$$\mu(K) = g_X - 1 \geq \mu(K') \Longrightarrow \mu_\alpha(T) \geq \mu_\alpha(T').$$

This is a contradiction. Hence T is α-semistable.

Assume that T is α-semistable. We will construct a α-orthogonal triple $F = (F_2 \xrightarrow{\psi} F_1)$ of T. This is done by the following lemma.

Lemma 4.2.4. *Let $T = (E_2 \xrightarrow{\varphi} E_1)$ be a triple on X of type (r_1, r_2, d_1, d_2) and $\alpha = \frac{a}{b} \in \mathbb{Q}$. If T is α-semistable then it has a α-orthogonal triple.*

Proof. Let E_T be the extension on $X \times \mathbb{P}^1$ defined by T. It follows from Theorem 3.3.4 that E_T is $H(\alpha)$-semistable of rank $r = r_1 + r_2$ and degree $b(d_1 + d_2) - 2ar_1$. By computing the Chern character up to numerical equivalence, we have

$$\mathrm{ch}(E_T) \equiv r_1 + r_1 + (d_1 + d_2)F_p - 2r_1F_q - 2d_1[pt].$$

Therefore

$$\Delta(E_T) = 2\,\mathrm{ch}_0(E_T)\,\mathrm{ch}_2(E_T) - \mathrm{ch}_1^2(E_T) = 4(r_1d_2 - r_2d_1). \tag{4.15}$$

Let us fix $C \in |m_0H(\alpha)|$ a smooth projective curve where

$$m_0 = \lceil \frac{1-r}{r}\Delta(E_T) + 1 \rceil = \lceil \frac{4(1 - (r_1 + r_2))(d_2r_1 - d_1r_2)}{r_1 + r_2} + 1 \rceil. \tag{4.16}$$

Here we denote by $\lceil x \rceil$ the smallest integral number which is not less than x for any $x \in \mathbb{Q}$. Applying Corollary 2.3.4 for E_T, we see that $E_T|_C$ is again a semistable vector bundle of rank r and degree

$$d = m_0 \deg_{H(\alpha)}(E_T) = bm_0\mu_\alpha(T).$$

By Theorem 4.1.3, there exists a vector bundle F_C on C such that $H^*(C, E_T|_C \otimes F_C) = 0$. Let f be the composition $C \hookrightarrow X \times \mathbb{P}^1 \longrightarrow X$ which is a finite morphism. Consider the following diagram,

$$\begin{tikzcd}
0 \arrow[r] & p^*E_1 \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-2) \arrow[r] \arrow[d, equal] & E_T \arrow[r] \arrow[d] & p^*E_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & p^*E_1 \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-2) \arrow[r] & p^*E_1 \otimes V \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1) \arrow[r] & p^*E_1 \arrow[r] & 0
\end{tikzcd}$$

By restricting the above diagram to C and then twisting with F_C, the square on the left hand side becomes

$$\begin{tikzcd}
E_T|_C \otimes F_C \arrow[r] \arrow[d] & p^*E_2|_C \otimes F_C \arrow[d] \\
{[p^*E_1 \otimes V \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1)]|_C \otimes F_C} \arrow[r] & p^*E_1|_C \otimes F_C
\end{tikzcd}$$

We take the pushforward of this diagram to X by f and set

$$K = f_*(E_T|_C \otimes F_C), F_1 = f_*F_C, F_2 = f_*((V \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1))|_C \otimes F_C).$$

As f is finite,

$$R^1 f_*([p^*E_1 \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-2)]_C \otimes F_C) = 0.$$

So we get a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & f_*([p^*E_1 \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-2)]|_C \otimes F_C) \arrow[r] \arrow[d, equal] & K \arrow[r] \arrow[d] & E_2 \otimes F_1 \arrow[r] \arrow[d, "\varphi\otimes \mathrm{Id}_{F_1}"] & 0 \\
0 \arrow[r] & f_*([p^*E_1 \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-2)]|_C \otimes F_C) \arrow[r] & E_1 \otimes F_2 \arrow[r, "\mathrm{Id}_{F_1}\otimes\psi"] & E_1 \otimes F_1 \arrow[r] & 0,
\end{tikzcd}$$

where $\psi := f_*(q^*e|_C \otimes \mathrm{Id}_{F_C}) : F_2 \longrightarrow F_1$ is surjective. It implies that K is the pullback of two morphisms

$$\varphi \otimes \mathrm{Id}_{F_1} : E_2 \otimes F_1 \longrightarrow E_1 \otimes F_1 \text{ and } \mathrm{Id}_{E_1} \otimes \psi : E_1 \otimes F_2 \longrightarrow E_1 \otimes F_1.$$

Equivalently, K is the kernel of $\varphi \times \psi$, the morphism of triple product

$$(E_2 \xrightarrow{\varphi} E_1) \times (F_2 \xrightarrow{\psi} F_1).$$

We then have a short exact sequence of vector bundles,

$$0 \longrightarrow K \longrightarrow E_2 \otimes F_1 \oplus E_1 \otimes F_2 \xrightarrow{\varphi\times\psi} E_1 \otimes F_1 \longrightarrow 0.$$

Since f is finite,

$$H^*(X, K) = H^*(X, f_*(E_T|_C \otimes F_C)) = H^*(C, E_T|_C \otimes F_C) = 0.$$

So $F = (F_2 \xrightarrow{\psi} F_1)$ will be the triple which we are looking for if the numerical invariants of F_1 and F_2 satisfy condition (ii) in Definition 4.2.2.

Now we compute the rank and degree of $F_1 = f_*(F_C)$. Let us set

$$h = \gcd(r, d), r_0 = \frac{r}{h}, d_0 = \frac{d}{h}.$$

As we have seen,

$$\operatorname{rank} F_C = pr_0, \deg(F_C) = p(r_0(g_C - 1) - d_0) \tag{4.17}$$

for some $p \in \mathbb{Z}_{>0}$. Let G be a vector bundle on C of rank R and degree D. Since $C \in |m_0H(\alpha)|$ where $H(\alpha) = aF_p + bF_q$, $f : C \longrightarrow X$ is finite of degree m_0b. For any $x \in X$,

$$\dim(f_*G \otimes k(x)) = m_0bR.$$

Hence f_*G is a vector bundle of rank m_0bR. As f is finite, we have

$$\dim H^*(C,G) = \dim H^*(X, f_*G).$$

So Riemann-Roch formula implies that $\chi(f_*G) = \chi(G)$. But

$$\chi(f_*G) = \chi(G) \Longleftrightarrow \deg(f_*G) + \operatorname{rank}(f_*G)(1-g_X) = D + R(1-g_C).$$

it implies that $\deg(f_*G) = RB + D$ where $B = m_0b(g_X-1)+1-g_C$. Now we have,

$$\begin{aligned}
\operatorname{rank}(F_1) &= m_0bpr_0,\\
\operatorname{rank}(F_2) &= 2m_0bpr_0 = 2\operatorname{rank}(F_1),\\
\deg(F_1) &= pr_0(m_0b(g_X-1)+1-g_C) + p(r_0(g_C-1)-d_0)\\
&= m_0bpr_0(g_X-1) - pd_0,\\
\deg(F_2) &= 2pr_0(m_0b(g_X-1)+1-g_C) + \deg(V \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1))|_C \otimes F_C)\\
&= 2pr_0(m_0b(g_X-1)+1-g_C) - 2kr_0m_0a + 2p(r_0(g_C-1)-d_0)\\
&= 2(m_0bpr_0(g_X-1)-pd_0) - 2pr_0m_0a\\
&= 2\deg(F_1) - 2\alpha\operatorname{rank}(F_1).
\end{aligned}$$

It follows that

$$\operatorname{rank}(F_2) = 2\operatorname{rank}(F_1), \mu(F_2) = \mu(F_1) - \alpha.$$

□

The theorem is then completely proven. □

4.2.3 Determinant line bundles for triples

Let S be a connected scheme of finite type over $k, p : S \times X \longrightarrow S$ and $q : S \times X \longrightarrow X$ be the projections. Let $\mathcal{T} = (\mathcal{E}_2 \xrightarrow{\varphi_S} \mathcal{E}_1)$ be a family of triples on X parametrized by S. For any triple $F = (F_2 \xrightarrow{\psi} F_1)$ on X, the triple product of $\mathcal{T}$ and q^*F,

$$\mathcal{T} \times q^*F = (\mathcal{E}_2 \otimes q^*F_1 \oplus \mathcal{E}_1 \otimes q^*F_2 \xrightarrow{\varphi_S \times q^*\psi} \mathcal{E}_1 \otimes q^*F_1),$$

is again a S-family of triples.

Definition 4.2.5. *Let $\mathcal{T} = (\mathcal{E}_2 \xrightarrow{\varphi_S} \mathcal{E}_1)$ be a family of triples on X parametrized by S and $F = (F_2 \xrightarrow{\psi} F_1)$ a triple on X. Then the line bundle*

$$\mathcal{L}_{\mathcal{T}\times q^*F} := \lambda_{\mathcal{E}_1}([F_2]) \otimes \lambda_{\mathcal{E}_2}([F_1]) \otimes (\lambda_{\mathcal{E}_1}([F_1]))^{-1},$$

where $\lambda_{\mathcal{E}_i}$ are morphisms defined in (4.1), *is called the determinant line bundle associated to $\mathcal{T}$ at F. As $\mathcal{T}$ is fixed, we denote $\mathcal{L}_{\mathcal{T}\times q^*F}$ just by $\mathcal{L}_{F_2 \longrightarrow F_1}$.*

We will show that the determinant line bundle associated to $\mathcal{T}$ at F defined above has similar properties of the determinant line bundle associated to a family of vector bundles as in Lemma 4.1.1: let $\mathcal{T} = (\mathcal{E}_2 \xrightarrow{\varphi_S} \mathcal{E}_1), \mathcal{T}' = (\mathcal{E}'_2 \xrightarrow{\varphi'_S} \mathcal{E}'_1)$ and $\mathcal{T}'' = (\mathcal{E}''_2 \xrightarrow{\varphi''_S} \mathcal{E}''_1)$ be S-families of triples on X such that the following sequence

$$0 \longrightarrow \mathcal{T}' \longrightarrow \mathcal{T} \longrightarrow \mathcal{T}'' \longrightarrow 0$$

is exact. Since the triple product is exact, the sequence

$$0 \longrightarrow \mathcal{T}' \times q^*F \longrightarrow \mathcal{T} \times q^*F \longrightarrow \mathcal{T}'' \times q^*F \longrightarrow 0$$

is exact. Precisely, we have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{E}'_2 \otimes q^*F_1 \oplus \mathcal{E}'_1 \otimes q^*F_2 \arrow[r] \arrow[d] & \mathcal{E}_2 \otimes q^*F_1 \oplus \mathcal{E}_1 \otimes q^*F_2 \arrow[r] \arrow[d] & \mathcal{E}''_2 \otimes q^*F_1 \oplus \mathcal{E}''_1 \otimes q^*F_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{E}'_1 \otimes q^*F_1 \arrow[r] & \mathcal{E}_1 \otimes q^*F_1 \arrow[r] & \mathcal{E}''_1 \otimes q^*F_1 \arrow[r] & 0,
\end{tikzcd} \tag{4.18}$$

where the rows are exact. From the exactness of the rows in (4.18) we have

$$\lambda_{\mathcal{E}_1}([F_2]) \otimes \lambda_{\mathcal{E}_2}([F_1]) \cong \lambda_{\mathcal{E}'_1}([F_2]) \otimes \lambda_{\mathcal{E}'_2}([F_1]) \otimes \lambda_{\mathcal{E}''_1}([F_2]) \otimes \lambda_{\mathcal{E}''_2}([F_1]),$$

$$\lambda_{\mathcal{E}_1}([F_1]) \cong \lambda_{\mathcal{E}'_1}([F_1]) \otimes \lambda_{\mathcal{E}''_1}([F_1]).$$

It is implied by the definition that

$$\begin{aligned}
\mathcal{L}_{\mathcal{T} \times q^*F} &:= \lambda_{\mathcal{E}_1}([F_2]) \otimes \lambda_{\mathcal{E}_2}([F_1]) \otimes (\lambda_{\mathcal{E}_1}([F_1]))^{-1} \\
&\cong \lambda_{\mathcal{E}'_1}([F_2]) \otimes \lambda_{\mathcal{E}'_2}([F_1]) \otimes \lambda_{\mathcal{E}''_1}([F_2]) \otimes \lambda_{\mathcal{E}''_2}([F_1]) \otimes \lambda_{\mathcal{E}'_1}([F_1])^{-1} \otimes \lambda_{\mathcal{E}''_1}([F_1])^{-1} \\
&\cong [\lambda_{\mathcal{E}'_1}([F_2]) \otimes \lambda_{\mathcal{E}'_2}([F_1]) \otimes \lambda_{\mathcal{E}'_1}([F_1])^{-1}] \otimes [\lambda_{\mathcal{E}''_1}([F_2]) \otimes \lambda_{\mathcal{E}''_2}([F_1]) \otimes \lambda_{\mathcal{E}''_1}([F_1])^{-1}] \\
&\cong \mathcal{L}_{\mathcal{T}' \times q^*F} \otimes \mathcal{L}_{\mathcal{T}'' \times q^*F}.
\end{aligned}$$

Similarly, if $0 \longrightarrow F' \longrightarrow F \longrightarrow F'' \longrightarrow 0$ is an exact sequence of triples, we have

$$\mathcal{L}_{\mathcal{T} \times q^*F} \cong \mathcal{L}_{\mathcal{T} \times q^*F'} \otimes \mathcal{L}_{\mathcal{T} \times q^*F''}.$$

Using the distributive property of pullback over tensor product, the base-change property is also fulfilled, i.e. if $f : S' \longrightarrow S$ be a morphism of scheme then

$$f^*\mathcal{L}_{\mathcal{T} \times q^*F} = \mathcal{L}_{f_X^*\mathcal{T} \times q'^*F},$$

where $f_X = f \times \mathrm{Id}_X : S' \times X \longrightarrow S \times X$ and $q' : S' \times X \longrightarrow X$ is the second projection. For any line bundle L on S, we have

$$\mathcal{T} \otimes p^*L = (\mathcal{E}_2 \otimes p^*L \xrightarrow{\varphi_S \times p^* \mathrm{Id}_L} \mathcal{E}_1 \otimes p^*L)$$

and then

$$\begin{aligned}\mathcal{L}_{\mathcal{T}\otimes p^*L\times q^*F} &= \lambda_{\mathcal{E}_1\otimes p^*L}([F_2])\otimes\lambda_{\mathcal{E}_2\otimes p^*L}([F_1])\otimes(\lambda_{\mathcal{E}_1\otimes p^*L}([F_1]))^{-1}\\ &= \mathcal{L}_{\mathcal{T}\times q^*F}\otimes L^{\chi(\mathcal{E}_{1s}\otimes F_2)+\chi(\mathcal{E}_{2s}\otimes F_1)-\chi(\mathcal{E}_{1s}\otimes F_1)}\\ &= \mathcal{L}_{\mathcal{T}\times q^*F}\otimes L^{\chi(\mathcal{E}_s\times F)}\end{aligned}$$

for some $s\in S$.

Since $\lambda_{\mathcal{E}_i}([F])$ depends only on the class $[F]\in K(X)$ (see (4.9)), we get the following similar property for determinant line bundles for triples.

Lemma 4.2.6. *Let* $\mathcal{T}=(\mathcal{E}_2\xrightarrow{\varphi_S}\mathcal{E}_1)$ *be a family of triples on* X *parametrized by* S *and* $F=(F_2\xrightarrow{\psi}F_1), F'=(F_2'\xrightarrow{\psi'}F_1')$ *be triples on* X*. If* $\operatorname{rank}(F_i')=\operatorname{rank}(F_i)$ *and* $\det(F_i')=\det(F_i)$ *for* $i=1,2$ *then*

$$\mathcal{L}_{\mathcal{T}\times q^*F'}\cong\mathcal{L}_{\mathcal{T}\times q^*F}.$$

4.2.4 Invariant sections

Let $F=(F_2\xrightarrow{\psi}F_1)$ be a triple on X where ψ is surjective and $\mathcal{T}=(\mathcal{E}_2\xrightarrow{\varphi_S}\mathcal{E}_1)$ be a family of triples on X parametrized by S. By taking the triple product, we have an exact sequence of vector bundles on $S\times X$,

$$0\longrightarrow\mathcal{K}_F\longrightarrow\mathcal{E}_2\otimes q^*F_1\oplus\mathcal{E}_1\otimes q^*F_2\longrightarrow\mathcal{E}_1\otimes q^*F_1\longrightarrow 0. \tag{4.19}$$

Hence

$$\mathcal{L}_{F_2\twoheadrightarrow F_1}:=\lambda_{\mathcal{E}_1}([F_2])\otimes\lambda_{\mathcal{E}_2}([F_1])\otimes(\lambda_{\mathcal{E}_1}([F_1]))^{-1}\cong\lambda_{\mathcal{K}_F}([\mathcal{O}_X]), \tag{4.20}$$

where $\lambda_{\mathcal{K}_F}([\mathcal{O}_X])$ is the determinant line bundle associated to S-family of vector bundles $\mathcal{K}_F$. Let us set $\theta_{F_2\twoheadrightarrow F_1}:=\mathcal{L}^{-1}_{F_2\twoheadrightarrow F_1}$. Then $\theta_{F_2\twoheadrightarrow F_1}\cong\lambda_{\mathcal{K}_F}([\mathcal{O}_X])^{-1}$. For any $s\in S$, we have an exact sequence of vector bundles on X

$$0\longrightarrow(\mathcal{K}_F)_s\longrightarrow\mathcal{E}_{2s}\otimes F_1\oplus\mathcal{E}_{1s}\otimes F_2\longrightarrow\mathcal{E}_{1s}\otimes F_1\longrightarrow 0.$$

It follows that $\chi((\mathcal{K}_F)_s)=\chi(\mathcal{E}_s\times F)$. As we have seen, if $\chi((\mathcal{K}_F)_s)=0$ for all $s\in S$, the line bundle $\lambda_{\mathcal{K}_F}([\mathcal{O}_X])^{-1}$ has a geometric section, denoted by $\zeta_{\mathcal{K}_F}$. Hence $\theta_{F_2\twoheadrightarrow F_1}$ also has a geometric section, denoted by $\zeta_{F_2\twoheadrightarrow F_1}$, via the isomorphism $\theta_{F_2\twoheadrightarrow F_1}\cong\lambda_{\mathcal{K}_F}([\mathcal{O}_X])^{-1}$. The zero divisor of $\zeta_{F_2\twoheadrightarrow F_1}$, denoted by $\Theta_{F_2\twoheadrightarrow F_1}$, parametrizes the same points as the zero divisor of $\zeta_{\mathcal{K}_F}$,

$$\Theta_{F_2\twoheadrightarrow F_1}=\{s\in S|H^0(X,(\mathcal{K}_F)_s)\neq 0\}. \tag{4.21}$$

Let $T=(E_2\xrightarrow{\varphi}E_1)$ be any triple of type (r_1,r_2,d_1,d_2) and $\alpha=\frac{a}{b}$ be a rational number. Let E_T be the extension bundle of rank $r=r_1+r_2$ defined by T and $m_0=\lceil\frac{1-r}{r}\Delta(E_T)+1\rceil$.

Let us set $d = m_0 \deg_{H(\alpha)}(E_T)$ and $h = \gcd(r, d)$. For each positive integer p, let Comp(p) be the set of triples $F = (F_2 \overset{\psi}{\twoheadrightarrow} F_1)$ which have the following numerical invariants

$$\begin{aligned} &\operatorname{rank}(F_1) = pm_0br_0, \deg(F_1) = p(m_0br_0(g_X - 1) - d_0), \\ &\operatorname{rank}(F_2) = 2\operatorname{rank}(F_1), \deg(F_2) = 2\deg(F_1) - \alpha\operatorname{rank}(F_1). \end{aligned} \tag{4.22}$$

where $r_0 = \frac{r}{h}, d_0 = \frac{d}{h}$. These triples can be constructed as in the proof of Lemma 4.2.4. Let us set

$$\mathrm{Comp} = \cup_{p\in\mathbb{Z}_{>0}} \mathrm{Comp}(p). \tag{4.23}$$

In particular, for any $F \in$ Comp and any triple T of type (r_1, r_2, d_1, d_2), we have

$$\begin{aligned} \chi(T \times F) &= \chi(E_2 \otimes F_1 \oplus E_1 \otimes F_2) - \chi(E_1 \otimes F_1) \\ &= (d_1 + d_2 - 2\alpha r_1)\operatorname{rank}(F_1) + \deg(F_1)(r_1 + r_2) + r_{F_1}(r_1 + r_2)(1 - g_X) \\ &= (d_1 + d_2 - 2\alpha r_1)m_0bpr_0 + r(m_0bpr_0(g_X - 1) - pd_0) + rm_0bpr_0(1 - g_X) \\ &= (d_1 + d_2 - 2\alpha r_1)m_0bpr_0 - rpd_0 = 0 \end{aligned} \tag{4.24}$$

since $m_0b(d_1 + d_2 - 2\alpha r_1) = m_0 \deg_{H(\alpha)}(E_T)$.

Assume that $\mathcal{T} = (\mathcal{E}_2 \xrightarrow{\varphi_S} \mathcal{E}_1)$ is a S-family of triples of type (r_1, r_2, d_1, d_2). By the definition of α-orthogonal triples, for any $F = (F_2 \overset{\psi}{\twoheadrightarrow} F_1) \in$ Comp, the zero divisor of geometric section $\zeta_{F_2 \twoheadrightarrow F_1}$ is described as follows

$$\Theta_{F_2\twoheadrightarrow F_1} = \{s \in S | F \text{ is not a } \alpha\text{-orthogonal triple of } \mathcal{T}_s\}. \tag{4.25}$$

4.3 Base point freeness

4.3.1 Generalized theta line bundles for triples

Let $T = (E_2 \xrightarrow{\varphi} E_1)$ be a α-semistable triple of type (r_1, r_2, d_1, d_2) on X where $\alpha = \frac{a}{b}$ and E_T be the extension on $X \times \mathbb{P}^1$ defined by T. We know that E_T is $H(\alpha)$-semistable. Moreover, the restriction of E_T to smooth projective curve $C' \in |mH(\alpha)|$ is still semistable for any $m \geq m_0 = \lceil \frac{1-r}{r}\Delta(E_T) + 1\rceil$. It is easy to see that T and T' are S-equivalent α-semistable triples if and only if E_T and $E_{T'}$, the extensions defined by T and T', respectively, are S-equivalent. We will show that for $m \gg m_0$, the S-equivalence is preserved under the restriction to a smooth projective curve $C' \in |mH(\alpha)|$.

Lemma 4.3.1. *Let E_T and $E_{T'}$ be $H(\alpha)$-semistable vector bundles on $X \times \mathbb{P}^1$ defined by triples T and T', respectively. Then for m large enough, $E_T|_{C'}$ and $E_{T'}|_{C'}$ are S-equivalent if and only if T and T' are S-equivalent.*

Proof. Assume that
$$0 \subset T^1 \subset T^2 \subset ... \subset T^n = T$$
is a Jordan-Höder filtration of T. By the construction of extensions, it follows that
$$0 \subset E_{T^1} \subset E_{T^2} \subset ... \subset E_{T^n} = E_T$$
is a Jordan-Höder filtration of E_T. Since $E_{T^i}/E_{T^{i-1}}$ are stable of $H(\alpha)$-slope $\mu_{H(\alpha)}(E_T)$ for all $i \geq 1$, for any $m \geq m_0$, $E_{T^i}|_{C'}/E_{T^{i-1}}|_{C'}$ are stable of the same slope (see also the proof of Corollary 2.3.4) since
$$E_{T^i}|_{C'}/E_{T^{i-1}}|_{C'} \cong (E_{T^i}/E_{T^{i-1}})|_{C'}.$$
Hence
$$0 \subset E_{T^1}|_{C'} \subset E_{T^2}|_{C'} \subset ... \subset E_{T^n}|_{C'} = E_T|_{C'}$$
is again a Jordan-Höder filtration of $E_T|_{C'}$. So we have
$$\mathrm{gr}(E_T|_{C'}) \cong \mathrm{gr}(E_T)|_{C'}.$$
It implies that $\mathrm{gr}(E_T|_C) \cong \mathrm{gr}(E_{T'}|_C)$ if $\mathrm{gr}(T) \cong \mathrm{gr}(T')$.

Conversely, it is enough to show that for m large enough, if E and E' are $H(\alpha)$-stable vector bundles on $X \times \mathbb{P}^1$ of the same slope such that $E|_{C'}$ and $E'|_C$ are isomorphic and stable then $E \cong E'$. Since E and E' are stable of the same slope, then
$$E \cong E' \Longleftrightarrow H^0(X \times \mathbb{P}^1, E^\vee \otimes E') \neq 0.$$
Consider the following exact sequence on $X \times \mathbb{P}^1$,
$$0 \longrightarrow E^\vee \otimes E' \otimes \mathcal{O}_{X\times\mathbb{P}^1}(-mH(\alpha)) \longrightarrow E^\vee \otimes E' \longrightarrow (E^\vee \otimes E')|_{C'} \longrightarrow 0.$$
We then obtain a long exact sequence of cohomology
$$\begin{aligned} 0 &\longrightarrow H^0(X \times \mathbb{P}^1, E^\vee \otimes E' \otimes \mathcal{O}_{X\times\mathbb{P}^1}(-mH(\alpha))) \longrightarrow H^0(X \times \mathbb{P}^1, E^\vee \otimes E') \\ &\longrightarrow H^0(X \times \mathbb{P}^1, (E^\vee \otimes E')|_{C'}) \longrightarrow H^1(X \times \mathbb{P}^1, E^\vee \otimes E' \otimes \mathcal{O}_{X\times\mathbb{P}^1}(-mH(\alpha))) \end{aligned}$$
where $H^0(X \times \mathbb{P}^1, (E^\vee \otimes E')|_{C'}) \cong H^0(C', E^\vee|_{C'} \otimes E'|_{C'}) \neq 0$ since $E|_{C'} \cong E'|_{C'}$. We have $mH(\alpha) = maF_p + mbF_q$ and $a, b > 0$. As
$$E = E_T \in \mathrm{Ext}^1(p^*E_1 \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-2), p^*E_2), E' = E_{T'} \in \mathrm{Ext}^1(p^*E_1' \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-2), p^*E_2'),$$
by Künneth formula, the vanishing of
$$H^0(X \times \mathbb{P}^1, E^\vee \otimes E' \otimes \mathcal{O}_{X\times\mathbb{P}^1}(-mH(\alpha))) \text{ and } H^1(X \times \mathbb{P}^1, E^\vee \otimes E' \otimes \mathcal{O}_{X\times\mathbb{P}^1}(-mH(\alpha)))$$

is induced from the vanishing of

$$H^0(X, E_j^\vee \otimes E_i'(-ma)) \text{ and } H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-mb+2))$$

which is satisfied for $m \gg m_0$. It follows that

$$H^0(X \times \mathbb{P}^1, E^\vee \otimes E') \cong H^0(X \times \mathbb{P}^1, (E^\vee \otimes E')|_{C'}) \neq 0.$$

Hence $E \cong E'$. □

This lemma implies that for any $m \geq m_0$, we have a well defined morphism between moduli spaces,

$$\begin{aligned} \mathfrak{t}_m : M_\alpha^{ss}(\underline{r}, \underline{d}) &\longrightarrow U_{C'}(r, d') \\ T &\mapsto E_T|_{C'}. \end{aligned} \tag{4.26}$$

and $\mathfrak{t}_m$ is an injective if $m \gg m_0$.

Let F_C be a complementary vector bundle of $U_C(r, d)$ where $C \in |m_0 H(\alpha)|$. Then the generalized theta line bundle θ_{F_C} has a section ζ_{F_C} whose zero divisor is Θ_{F_C}. Let us set

$$\mathfrak{t} = \mathfrak{t}_{m_0} : M_\alpha^{ss}(\underline{r}, \underline{d}) \longrightarrow U_C(r, d).$$

Definition 4.3.2. *We call the pullback $\mathfrak{t}^*\theta_{F_C}, \mathfrak{t}^*\zeta_{F_C}$ again a generalized theta line bundle and a generalized theta function, respectively, on $M_\alpha^{ss}(\underline{r}, \underline{d})$.*

4.3.2 Linear systems

Assume that $\mathcal{T} = (\mathcal{E}_2 \xrightarrow{\varphi_S} \mathcal{E}_1)$ is a family of α-semistable triples of type (r_1, r_2, d_1, d_2) on X parametrized by a connected scheme S. For any $F = (F_2 \overset{\psi}{\twoheadrightarrow} F_1) \in \mathrm{Comp}$, the line bundle $\theta_{F_2 \twoheadrightarrow F_1}$ has a geometric section $\zeta_{F_2 \twoheadrightarrow F_1}$ whose zero divisor is

$$\Theta_{F_2 \twoheadrightarrow F_1} = \{s \in S | F = (F_2 \overset{\psi}{\twoheadrightarrow} F_1) \text{ is not a } \alpha - \text{orthogonal triple of } \mathcal{T}_s\}.$$

For simplicity, we denote by $\mathcal{M}$ the moduli space $\mathcal{M}_\alpha^{ss}(\underline{r}, \underline{d})$. Consider the following diagram

$$S \xrightarrow{f_{\mathcal{T}}} \mathcal{M} \xrightarrow{\mathfrak{t}} U_C(r, d)$$

where $f_{\mathcal{T}}$ is induced from family $\mathcal{T}$. Let us recall the linear systems which exist on these schemes:

+ On $U_C(r, d)$: we have pluritheta linear series $|p\Theta_{F_0}|$ where Θ_{F_0} is a basic theta divisor. For any complementary vector bundle F_C such that $\det(F_C) \cong L_p$, the generalized theta line bundle θ_{F_C} has a section ζ_{F_C} whose zero divisor is Θ_{F_C}. Moreover, θ_{F_C} is isomorphic to $\theta_{F_0}^p$ for such any F_C.

+ On $\mathcal{M}$: we set $\mathcal{L} = \mathfrak{t}^*\theta_{F_0}$ and call it a basic theta line bundle. For each p, we have a generalized theta line bundle $\mathcal{L}^p$ and generalized theta functions $\mathfrak{t}^*(\zeta_{F_C})$ for any such complementary vector bundles F_C. Let $V(\mathcal{L}^p) \subset H^0(\mathcal{M}, \mathcal{L}^p)$ be the linear space spanned by sections $\mathfrak{t}^*\zeta_{F_C}$.
+ On S: Let us fix F_C a complementary vector bundle F_C of $U_C(r,d)$ as above, we consider the following triple constructed as in the proof of Lemma 4.2.4,

$$F = (F_2 \overset{\psi}{\twoheadrightarrow} F_1) := (f_*(V \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1)|_C \otimes F_C) \overset{\psi}{\twoheadrightarrow} f_*F_C), \tag{4.27}$$

where f is the composed morphism $C \hookrightarrow X \times \mathbb{P}^1 \longrightarrow X$ and $q : X \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ is the projection. It implies that $F \in \mathrm{Comp}(p)$ and then $\chi(\mathcal{E}_s \times F) = 0$ for any $s \in S$. Let $\mathrm{Comp}(p, F) \subset \mathrm{Comp}(p)$ be the subset of triples $F' = (F_2' \overset{\psi'}{\twoheadrightarrow} F_1') \in \mathrm{Comp}(p)$ such that $[F_i'] = [F_i] \in K(X), i = 1, 2$. It follows from Lemma 4.2.6 that all line bundles $\theta_{F_2' \twoheadrightarrow F_1'}$ are isomorphic to $\theta_{F_2 \twoheadrightarrow F_1}$. Hence we obtain a unique line bundle $\theta_{F_2 \twoheadrightarrow F_1}$ and a linear system $V(p, F) \subset H^0(S, \theta_{F_2 \twoheadrightarrow F_1})$ spanned by geometric sections $\zeta_{F_2' \twoheadrightarrow F_1'}$ where $F' = (F_2' \overset{\psi'}{\twoheadrightarrow} F_1') \in \mathrm{Comp}(p, F)$.

4.3.3 Corollaries

The separation property of $|p\Theta_{F_0}|$ on $U_C(r,d)$ implies that for any semistable vector bundle $E_C \in U_C(r,d)$ and any $p \geq p_0 := r^2 + r$, there exists a complementary vector bundle F_C of determinant L_p such that $\Theta_{F_C} \in |p\Theta_{F_0}|$ and $E_C \notin \mathrm{Supp}\,\Theta_{F_C}$. As a consequence, we obtain similar result for the base point freeness of the linear systems mentioned above on S and $\mathcal{M}$.

For any triple $F' = (F_2' \overset{\psi'}{\twoheadrightarrow} F_1') \in \mathrm{Comp}(p, F)$, we set

$$\overline{\Theta}_{F_2' \twoheadrightarrow F_1'} := \{T \in \mathcal{M} | F' = (F_2' \overset{\psi'}{\twoheadrightarrow} F_1') \text{ is not a } \alpha\text{-orthogonal triple of } T\}.$$

Corollary 4.3.3. *For any $p \geq p_0$, then*

(i) The linear systems $V(\mathcal{L}^p)$ and $V(p, F)$ are base point free.

(ii) There exists a triple $F' = (F_2' \overset{\psi'}{\twoheadrightarrow} F_1') \in \mathrm{Comp}(p, F)$ such that $\overline{\Theta}_{F_2' \twoheadrightarrow F_1'}$ is a Cartier divisor on $\mathcal{M}$ and

$$\mathcal{L}^p \cong \mathcal{O}_{\mathcal{M}}(\overline{\Theta}_{F_2' \twoheadrightarrow F_1'}).$$

Proof. Let T be any α-semistable triple on X. For the proof, we may assume that there exists $s \in S$ such that $\mathcal{E}_s = T$. Let E_T be the extension defined by T on $X \times \mathbb{P}^1$. Then $\mathfrak{t}(T) = E_T|_C \in U_C(r,d)$. Since $|p\Theta_{F_0}|$ is base point free, there exists a complementary vector bundle F'_C such that $\det(F'_C) \cong L_p, \Theta_{F'_C} \in |p\Theta_{F_0}|$ and $E_T|_C \notin \operatorname{Supp}\Theta_{F'_C}$. It implies that $\mathfrak{t}^*(\zeta_{F'_C})(T) \neq 0$. Let

$$F' = (F'_2 \overset{\psi'}{\twoheadrightarrow} F'_1) := (f_*(V \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1)|_C \otimes F'_C) \overset{\psi'}{\twoheadrightarrow} f_*F'_C),$$

be the triple constructed from F'_C. Then $F' \in \mathrm{Comp}(p, F)$ and moreover, it is a α-orthogonal triple of $T = \mathcal{E}_s$. Therefore $\zeta_{F'_2 \twoheadrightarrow F'_1}(s) \neq 0$. This proves (i) and moreover implies that $\mathfrak{t}^*\Theta_{F'_C}$ is a Cartier divisor. In particular, we have

$$\begin{aligned} \mathfrak{t}^*\Theta_{F'_C} &= \{T \in \mathcal{M} | E_T|_C \in \Theta_{F'_C}\} \\ &= \{T \in \mathcal{M} | H^0(C, E_T|_C \otimes F'_C) \neq 0\} \\ &= \{T \in \mathcal{M} | F' = (F'_2 \overset{\psi'}{\twoheadrightarrow} F'_1) \text{ is not a } \alpha\text{-orthogonal triple of } T\} \\ &= \overline{\Theta}_{F'_2 \twoheadrightarrow F'_1}. \end{aligned} \tag{4.28}$$

It follows that

$$\mathcal{L}^p = \mathfrak{t}^*\theta^p_{F_0} \cong \mathfrak{t}^*\mathcal{O}_{U_C(r,d)}(\Theta_{F'_C}) \cong \mathcal{O}_{\mathcal{M}}(\overline{\Theta}_{F'_2 \twoheadrightarrow F'_1}).$$

□

By the construction of the extensions defined by triples, for each S-family of α-semistable triples $\mathcal{T} = (\mathcal{E}_2 \xrightarrow{\varphi_S} \mathcal{E}_1)$ we obtain a family $E_{\mathcal{T}} = (E_{\mathcal{T}_s})_{s \in S}$ of $H(\alpha)$- semistable extensions on $X \times \mathbb{P}^1$ parametrized by S. Restricting this family to C, we get a family of semistable vector bundles on C,

$$\mathcal{E} = (E_{\mathcal{T}_s}|_C)_{s \in S}.$$

We also have a commutative diagram

$$\begin{tikzcd} S \arrow[rr, "f_{\mathcal{E}}"] \arrow[dr, "f_{\mathcal{T}}"'] & & U_C(r,d) \\ & \mathcal{M} \arrow[ur, "\mathfrak{t}"'] & \end{tikzcd} \tag{4.29}$$

where $f_{\mathcal{E}}$ is the morphism induced by $\mathcal{E}$. For any complementary vector bundle F_C of $U_C(r, d)$, it is implied by Theorem 4.1.2 that the line bundle $\lambda_{\mathcal{E}}([F_C])^{-1}$ descends to θ_{F_C} by $f_{\mathcal{E}}$. We obtain a similar universal property for the generalized theta line bundles on $\mathcal{M}$.

Corollary 4.3.4. *Let $p \geq 1$ and $F'' = (F''_2 \overset{\psi''}{\twoheadrightarrow} F''_1)$ be any triple in* $\mathrm{Comp}(p, F)$*. Then for any family $\mathcal{T} = (\mathcal{E}_2 \xrightarrow{\varphi_S} \mathcal{E}_1)$ of α-semistable triples on X parametrized by S, the line bundle $\theta_{F''_2 \twoheadrightarrow F''_1}$ descends to $\mathcal{L}^p$.*

Proof. Since $f_{\mathcal{E}} = \mathfrak{t} \circ f_{\mathcal{T}}$, $\lambda_{\mathcal{E}}([F_C])^{-1} \cong f_{\mathcal{T}}^* \mathcal{L}^p$ for any complementary vector bundle F_C such that $\det(F_C) = L_p$. Consider the following diagram

$$\begin{tikzcd}
 & S \times C \arrow[rr, "q_C"] \arrow[dl, "p_C"'] \arrow[dd, "f_S"] \arrow[dr, hook, "i_S"] & & C \arrow[dd, "f"] \arrow[dr, hook, "i"] & & \\
S & & S \times X \times \mathbb{P}^1 \arrow[rr, "p_{23}"] \arrow[dl, "p_{12}"] & & X \times \mathbb{P}^1 \arrow[r, "q"] \arrow[dl, "p"] & \mathbb{P}^1. \\
 & S \times X \arrow[ul, "p_X"] \arrow[rr, "q_X"] & & X & &
\end{tikzcd} \tag{4.30}$$

By the construction of $E_{\mathcal{T}}$, we have an exact sequence

$$0 \longrightarrow E_{\mathcal{T}} \longrightarrow p_{12}^*\mathcal{E}_2 \oplus p_{12}^*\mathcal{E}_1 \otimes V \otimes (q \circ p_{23})^* \mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow p_{12}^*\mathcal{E}_1 \longrightarrow 0.$$

Restricting to $S \times C$, we have

$$0 \longrightarrow \mathcal{E} \longrightarrow p_{12}^*\mathcal{E}_2|_{S\times C} \oplus p_{12}^*\mathcal{E}_1 \otimes V \otimes (q \circ p_{23})^* \mathcal{O}_{\mathbb{P}^1}(-1)|_{S\times C} \longrightarrow p_{12}^*\mathcal{E}_1|_{S\times C} \longrightarrow 0$$

(see (3.12) and (3.11)). By twisting with $q_C^* F_C$, we pass to the following exact sequence

$$\begin{aligned} 0 \longrightarrow \mathcal{E} \otimes q_C^* F_C \longrightarrow p_{12}^*\mathcal{E}_2|_{S\times C} \otimes q_C^* F_C \oplus p_{12}^*\mathcal{E}_1 \otimes V \otimes (q \circ p_{23})^* \mathcal{O}_{\mathbb{P}^1}(-1)|_{S\times C} \otimes q_C^* F_C \\ \longrightarrow p_{12}^*\mathcal{E}_1|_{S\times C} \otimes q_C^* F_C \longrightarrow 0. \end{aligned} \tag{4.31}$$

Since f is finite morphism, $(f_S)_*$ is an exact functor. We then obtain a short exact sequence on $S \times X$,

$$\begin{aligned} &0 \longrightarrow (f_S)_*(\mathcal{E} \otimes q_C^* F_C) \\ &\quad \longrightarrow (f_S)_*(p_{12}^*\mathcal{E}_2|_{S\times C} \otimes q_C^* F_C) \oplus (f_S)_*(p_{12}^*\mathcal{E}_1 \otimes V \otimes (q \circ p_{23})^* \mathcal{O}_{\mathbb{P}^1}(-1)|_{S\times C} \otimes q_C^* F_C) \\ &\quad \longrightarrow (f_S)_*(p_{12}^*\mathcal{E}_1|_{S\times C} \otimes q_C^* F_C) \longrightarrow 0. \end{aligned} \tag{4.32}$$

By projection formula and flat base change, we obtain the following exact sequence

$$0 \longrightarrow (f_S)_*(\mathcal{E} \otimes q_C^* F_C) \longrightarrow \mathcal{E}_2 \otimes q_X^* F_1 \oplus \mathcal{E}_1 \otimes q_X^* F_2 \longrightarrow \mathcal{E}_1 \otimes q_X^* F_1 \longrightarrow 0.$$

Since $p_C = p_X \circ f_S$ and $(f_S)_*$ is exact,

$$\theta_{F_2 \twoheadrightarrow F_1} \cong \lambda_{(f_S)_*(\mathcal{E} \otimes q_C^* F_C)}([\mathcal{O}_X])^{-1} \cong \lambda_{\mathcal{E}}([F_C])^{-1}.$$

Now for any $F'' = (F_2'' \overset{\psi''}{\twoheadrightarrow} F_1'') \in \mathrm{Comp}(p, F)$,

$$\theta_{F_2'' \twoheadrightarrow F_1''} \cong \theta_{F_2 \twoheadrightarrow F_1} \cong f_{\mathcal{T}}^* \mathcal{L}^p.$$

$\square$

We consider the numerical equivalence on $K(X \times \mathbb{P}^1)$ which generalizes the numerical equivalence of divisors defined in Chapter 2 (cf. [12, §8, p.178]). Let u and u' be classes in $K(X \times \mathbb{P}^1)$. Then they are said to be numerically equivalent: $u \equiv u'$, if their difference is contained in the radical of the quadratic from $(a, b) \mapsto \chi(a \cdot b)$. Let

$$K(X \times \mathbb{P}^1)_{\mathrm{Num}} = K(X \times \mathbb{P}^1)/\equiv .$$

It is easy to see that for any triple T of fixed type (r_1, r_2, d_1, d_2), the corresponding extension E_T is contained in a fixed class $\mathfrak{e} \in K(X \times \mathbb{P}^1)_{\mathrm{Num}}$. Let $U_{X\times\mathbb{P}^1}(\mathfrak{e})$ be the moduli space of $H(\alpha)$-semistable vector bundles on $X \times \mathbb{P}^1$ of numerical class $\mathfrak{e}$.

Corollary 4.3.5. *The line bundle $\mathcal{L}$ on $\mathcal{M}$ is ample.*

Proof. It is enough to show that, $\mathcal{L}^p$ is ample for some $p \geq p_0$. Let F_C be a complementary vector bundle of $U_C(r, d)$ such that $\mathcal{L}^p = \mathfrak{t}^*\theta_{F_C}$. Let $C' \in |mC|$ where m large enough such that the morphism

$$\mathfrak{t}_m : \mathcal{M} \longrightarrow U_{C'}(r, md)$$

is an injection (see (4.26)). If we can find a complementary vector bundle $F_{C'}$ of $U_{C'}(r, md)$ such that $\mathfrak{t}_m^*\theta_{F_{C'}} \cong \mathcal{L}^{mp}$, we are done since $\mathfrak{t}_m^*\theta_{F_{C'}}$ is ample.

Assume that F_C has rank R and degree D. Consider the class $[F_C] \in K(X \times \mathbb{P}^1)$, we have

$$\mathrm{ch}([F_C]) = R\,\mathrm{ch}([\mathcal{O}_C]) + D = R.C + (D - \frac{RC^2}{2}).$$

Let K be the canonical divisor on $X \times \mathbb{P}^1$, then

$$g_C = \frac{1}{2}C^2 + \frac{1}{2}C.K + 1 \text{ and } g_{C'} = \frac{1}{2}C'^2 + \frac{1}{2}C'.K + 1.$$

As $C' \in |mC|$, we have

$$g_{C'} - 1 = m(g_C - 1) + \frac{m(m-1)}{2}C^2.$$

Let $F_{C'}$ be a vector bundle on C' of rank R and degree $D' = mD + \frac{m(m-1)}{2}RC^2$. We then have

$$rD' + Rmd + rR(1 - g_{C'}) = m(rD + Rd + rR(1 - g_C)) = 0,$$

i.e. $F_{C'}$ is a complementary vector bundle of $U_{C'}(r, md)$. Moreover,

$$\mathrm{ch}([\mathcal{O}_{C'}]) = m\,\mathrm{ch}([\mathcal{O}_C]) - \frac{m(m-1)}{2}C^2$$

and then

$$\mathrm{ch}([F_{C'}]) = R\,\mathrm{ch}([\mathcal{O}_{C'}]) + D' = R(m\,\mathrm{ch}([\mathcal{O}_C]) - \frac{m(m-1)}{2}C^2) + D' = m\,\mathrm{ch}([F_C]).$$

It implies that $[F_{C'}] = m[F_C] \in K(X \times \mathbb{P}^1)$.

Consider the following factors of $\mathfrak{t}$ and $\mathfrak{t}_m$

$$\begin{tikzcd}
 & U_C(r,d) \\
\mathcal{M} \arrow[ur, "\mathfrak{t}"] \arrow[r, hook] \arrow[dr, "\mathfrak{t}_m"'] & U_{X\times\mathbb{P}^1}(\mathfrak{e}) \arrow[u, "i^*"'] \arrow[d, "i_m^*"] \\
 & U_{C'}(r,md)
\end{tikzcd} \tag{4.33}$$

where i^* and i_m^* are restrictions induced from the inclusions

$$i : C \hookrightarrow X \times \mathbb{P}^1 \text{ and } i_m : C' \hookrightarrow X \times \mathbb{P}^1,$$

respectively. Since $[F_{C'}] = m[F_C]$, it implies that $i_m^*\theta_{F_{C'}} \cong (i^*\theta_{F_C})^m$. Therefore, we have

$$\mathfrak{t}_m^*\theta_{F_{C'}} \cong (\mathfrak{t}^*\theta_{F_C})^m \cong \mathcal{L}^{mp}.$$

□

4.4 Langton's valuative criterion for triples

4.4.1 Elementary transformations

Let R be a discrete valuation ring where $\operatorname{Spec} R = \{\eta, 0\}$ and $X_R = \operatorname{Spec} R \times X$. Let us denote by $X_\eta = \eta \times X$ the generic fiber and X_0 the closed fiber of the first projection

$$p_R : \operatorname{Spec} R \times X \longrightarrow \operatorname{Spec} R.$$

Then X_0 is a Cartier divisor on X_R and any vector bundle on X_0 is a sheaf on X_R of projective dimension 1. Let $\mathcal{E}$ be a vector bundle on X_R and E_0 be the restriction of $\mathcal{E}$ on closed fiber X_0. We recall now the construction of elementary transformations of $\mathcal{E}$ along a quotient bundle of E_0, from which we can similarly construct the elementary transformations of triples.

Let E be a vector bundle on X_0 which is a quotient of E_0. Then the kernel $\mathcal{E}'$ of the composition

$$\pi : \mathcal{E} \to E_0 \overset{\pi_0}{\twoheadrightarrow} E$$

is a vector bundle on X_R. We call it an elementary transformation of $\mathcal{E}$ along E. If $\mathcal{E}'$ is an elementary transformation of $\mathcal{E}$ along E, we have a short exact sequence of sheaves on X_R

$$0 \to \mathcal{E}' \to \mathcal{E} \overset{\pi}{\to} E \to 0. \tag{4.34}$$

By restricting this exact sequence to X_η, we see easily that $\mathcal{E}'$ and $\mathcal{E}$ coincide over the generic fiber. On closed fiber, E becomes a subbundle of $\mathcal{E}'$.

Similarly, let $\mathcal{T} = (\mathcal{E}_2 \xrightarrow{\varphi_R} \mathcal{E}_1)$ be a triple on X_R and $T = (E_2 \xrightarrow{\varphi} E_1)$ be a quotient of T_0, the restriction of $\mathcal{T}$ on the closed fiber X_0. Let $\phi : \mathcal{T} \twoheadrightarrow T$ be the composed morphism of triples on X_R.

Definition 4.4.1. *The triple $\mathcal{T}' = (\mathcal{E}'_2 \xrightarrow{\varphi'_R} \mathcal{E}'_1)$ is called an elementary transformation of $\mathcal{T}$ along T if it satisfies the following short exact sequence of triples*

$$0 \to \mathcal{T}' \to \mathcal{T} \xrightarrow{\phi} T \to 0. \tag{4.35}$$

Notice that, the sequence (4.35) is a short exact sequence of triples of sheaves on X_R. But $\mathcal{T}'$ is really a triple on X_R, i.e. $\mathcal{E}'_1$ and $\mathcal{E}'_2$ are vector bundle on X_R. As above, we can also conclude that the elementary transformations of $\mathcal{T}$ along T do not change the triple over the generic point. But they transfer the quotient T of T_0 to a subtriple of $\mathcal{T}'$ over the closed point.

Lemma 4.4.2. *Let $T = (E_2 \xrightarrow{\varphi} E_1)$ be a triple on X which is not α-semistable and $T' = (E'_2 \xrightarrow{\varphi'} E'_1)$ be the maximal destabilizing subtriple of T. Then the quotient*

$$T'' = T/T' := (E''_2 \xrightarrow{\varphi''} E''_1)$$

is again a triple, i.e. E''_i is vector bundle on X, $i = 1, 2$.

Proof. Let F'_i be the torsion subsheaf of $E''_i, i = 1, 2$. It is clear that $\varphi''(F'_2) \subset F'_1$. Then $F_i = E''_i/F'_i$ are vector bundles and $F = (F_2 \xrightarrow{\overline{\varphi''}} F_1)$ is a triple. We have a short exact sequence

$$0 \longrightarrow F' \longrightarrow T'' \longrightarrow F \longrightarrow 0$$

where $F' = (F'_2 \xrightarrow{\varphi''} F'_1)$. Let G be the kernel of the composed morphism $T' \twoheadrightarrow T'' \twoheadrightarrow F$. We obtain a short exact sequence

$$0 \longrightarrow T' \longrightarrow G \longrightarrow F' \longrightarrow 0.$$

It implies that,

$$\mu_\alpha(G) = \mu_\alpha(T') + \frac{\mathrm{length}(F'_1) + \mathrm{length}(F'_2)}{r'_1 + r'_2} \geq \mu_\alpha(T'),$$

where r'_i is the rank of $E'_i, i = 1, 2$. Since T' is maximal destabilizing subtriple of T, $T' = G$. It follows that $F'_i = 0$ for $i = 1, 2$. Hence E''_i are vector bundles. $\square$

4.4.2 Valuative criterion

Lemma 4.4.3 ([10, Lm. 6.1])**.** *Let X be a smooth projective curve. Any vector bundle E on X_η can be extended to a vector bundle $\mathcal{E}$ on X_R.*

In [10, Thm. 6.4], Hein gave a proof of Langton's theorem for semistable vector bundles of rank 2 and determinant ω_X using the elementary transformations and the cohomological characterization of semistability of vector bundles. By the same method, we can prove the following Langton's theorem for α-semistable triples(for the original Langton's valuative criterion, see [14]).

Theorem 4.4.4. *Let T_η be a α-semistable triple of type (r_1, r_2, d_1, d_2) on the generic fiber X_η. Then the exists a triple $\mathcal{T}$ on X_R such that $\mathcal{T}|_{X_\eta} \cong T_\eta$ and $T_0 := \mathcal{T}|_{X_0}$ is α-semistable.*

***Proof*.** Assume that $T_\eta = (E_{\eta 2} \xrightarrow{\varphi_\eta} E_{\eta 1})$ on X_η. It is implied from Lemma 4.4.3 that there exist vectors bundles $\mathcal{E}_2$ and $\mathcal{E}_1$ on X_R which extend $E_{\eta 2}$ and $E_{\eta 1}$, respectively. We consider the morphism $\varphi_\eta \in \mathrm{Hom}_{X_\eta}(E_{\eta 2}, E_{\eta 1})$ as a section of $\mathcal{E}_2^\vee \otimes \mathcal{E}_1$ over the open set X_η. By a standard argument as in [9, Lm. 5.14, p. 118], we can extend this section to obtain a section $\varphi_R \in \mathrm{Hom}_{X_R}(\mathcal{E}_2, \mathcal{E}_1)$. Hence we get an extension $\mathcal{T} = (\mathcal{E}_2 \xrightarrow{\varphi_R} \mathcal{E}_1)$ of T_η on X_R.

Let $p_R : X_R \longrightarrow \operatorname{Spec} R$ and $q_R : X_R \longrightarrow X$ be the projections. For any $F = (F_2 \overset{\psi}{\twoheadrightarrow} F_1) \in \mathrm{Comp}$, we take the triple product $\mathcal{T} \times q_R^* F$ and obtain a short exact sequence of vector bundles on X_R,

$$0 \longrightarrow \mathcal{K}_F \longrightarrow \mathcal{E}_2 \otimes q_R^* F_1 \oplus \mathcal{E}_1 \otimes q_R^* F_2 \longrightarrow \mathcal{E}_1 \otimes q_R^* F_1 \longrightarrow 0.$$

We now define an integer $\mathrm{bad}(\mathcal{T})$

$$\mathrm{bad}(\mathcal{T}) := \min_{F \in \mathrm{Comp}} \{\mathrm{bad}(\mathcal{K}_F))\}$$

where $\mathrm{bad}(\mathcal{K}_F)$ is the number introduced first by Hein in [10]: it is equal to ∞ if $\mathrm{Supp}(R^1 p_{R*} \mathcal{K}_F)$ is whole $\operatorname{Spec} R$ and to $\mathrm{length}(R^1 p_{R*} \mathcal{K}_F)$ otherwise.

Since T_η is α-semistable, it follows from Lemma 4.2.4 that there exists $F_\eta \in \mathrm{Comp}$ such that $H^1(X_\eta, K_\eta) = 0$ where K_η is the restrictions of $\mathcal{K}_{F_\eta}$ to X_η. Therefore $R^1 p_{R*} \mathcal{K}_{F_\eta}$ is zero at generic point η. So we have

$$\mathrm{bad}(\mathcal{T}) \leq \mathrm{length}(R^1 p_{R*} \mathcal{K}_{F_\eta}),$$

which is finite. It is also implied by Theorem 4.2.3 and base change that

$$T_0 \text{ is } \alpha\text{-semistable} \Longleftrightarrow \mathrm{bad}(\mathcal{T}) = 0.$$

Assume that T_0 is not α-semistable. Then $\mathrm{bad}(\mathcal{T})$ is positive. It is implied by Lemma 4.4.2 that there exists a quotient T of T_0 such that

$$\mu_\alpha(T) < \mu_\alpha(T_0).$$

Let $\mathcal{T}'$ be an elementary transformation of $\mathcal{T}$ along T. If we can show that

$$\mathrm{bad}(\mathcal{T}') < \ \mathrm{bad}(\mathcal{T})$$

then after taking a finite number of elementary transformations, we obtain the right one, i.e. an elementary transformation $\mathcal{T}'$ such that $\mathrm{bad}(\mathcal{T}')$ is zero.

For any $F \in \mathrm{Comp}$, we have (see (4.24))

$$\chi(T_s \times F) = \chi(K_s) = 0$$

where T_s and K_s are the restriction of $\mathcal{T}$ and $\mathcal{K}_F$, respectively, on the fiber over $s \in \operatorname{Spec} R$. Choose a triple $F = (F_2 \overset{\psi}{\twoheadrightarrow} F_1) \in \mathrm{Comp}$ such that

$$\mathrm{bad}(\mathcal{T}) = \mathrm{length}(R^1 p_{R*}\mathcal{K}_F).$$

It implies that $R^1 p_{R*}\mathcal{K}_F$ is supported only on the closed point $0 \in \operatorname{Spec} R$. So we have

$$H^0(X_\eta, K_\eta) = H^1(X_\eta, K_\eta) = 0.$$

Hence $p_{R*}\mathcal{K}_F = 0$ since it is torsion free and zero at generic point.

Consider the following exact sequence of triples on X_R,

$$0 \longrightarrow \mathcal{T}' \longrightarrow \mathcal{T} \longrightarrow T \longrightarrow 0.$$

By taking triple product with $q_R^* F$ and applying the kernel-cokernel sequence to the obtained diagram, we have the following exact sequence

$$0 \longrightarrow \mathcal{K}'_F \longrightarrow \mathcal{K}_F \longrightarrow K \longrightarrow 0.$$

Since $p_{R*}\mathcal{K}_F = 0$, by applying functor p_{R*} to this exact sequence, we get an exact sequence

$$0 \longrightarrow p_{R*}K \longrightarrow R^1 p_{R*}\mathcal{K}'_F \longrightarrow R^1 p_{R*}\mathcal{K}_F \longrightarrow R^1 p_{R*}K \longrightarrow 0.$$

We have $p_{R*}K = H^0(X_0, K)$ and $R^1 p_{R*}K = H^1(X_0, K)$ which are modules of finite length on R. Hence $R^1 p_{R*}\mathcal{K}'_F$ has finite length on R. In particular, we have

$$\mathrm{length}(R^1 p_{R*}\mathcal{K}'_F) = \mathrm{length}(R^1 p_{R*}\mathcal{K}_F) + \chi(K).$$

Assume that $T = (E_2 \xrightarrow{\varphi} E_1)$ on $X_0 \cong X$, then

$$0 \longrightarrow K \longrightarrow E_1 \otimes F_2 \oplus E_2 \otimes F_1 \longrightarrow E_1 \otimes F_1 \longrightarrow 0,$$

is an exact sequence. We have

$$\mu(K) = \mu_\alpha(T) + \mu(F_1)$$

(see also (4.14)). Similarly $\mu(K_0) = \mu_\alpha(T_0) + \mu(F_1)$. It implies that $\mu(K) < \mu(K_0)$ since $\mu_\alpha(T_0) > \mu_\alpha(T)$. So we have

$$\chi(K) < \chi(K_0) = 0$$

and then

$$\mathrm{length}(R^1 p_{R*}\mathcal{K}'_F) < \mathrm{length}(R^1 p_{R*}\mathcal{K}_F).$$

$\square$

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